

WAVES IN THE WITTEN BUBBLE OF NOTHING AND THE HAWKING WORMHOLE

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ABSTRACT. We investigate the propagation of the scalar waves in the Witten space-time called “*bubble of nothing*” and in its remarkable sub-manifold, the Lorentzian Hawking wormhole. Due to the global hyperbolicity, the global Cauchy problem is well-posed in the functional framework associated with the energy. We perform a complete spectral analysis that allows to get an explicit form of the solutions in terms of special functions. If the effective mass is non zero, the profile of the waves is asymptotically almost periodic in time. In contrast, the massless case is dispersive. We develop the scattering theory, classical as well as quantum. The quantized scattering operator leaves invariant the Fock vacuum: there is no creation of particles. The resonances can be defined in the massless case and they are purely imaginary.

I. INTRODUCTION

In 1982, E. Witten introduced a fascinating space-time in the framework of the quantum cosmology: he claimed in [54] that the Kaluza-Klein universe $(\mathbb{R}_\tau \times \mathbb{R}_\xi^3 \times S_\psi^1, ds^2 = d\tau^2 - d\xi^2 - d\psi^2)$ is quantum mechanically unstable, and he constructed an Euclidean instanton for the decay by the formation of a *bubble of nothing* that nucleates and expands exponentially fast, ever closer to the speed of light, eating up the entire spacetime. This work was a seminal step in quantum cosmology that is mainly interested in the quantum stability of the universe. The scenario of Witten has been extensively studied and various generalizations have since been found. Among numerous papers by the physicists we can cite the following works. Brill and Horowitz [14] proved in a similar way that nonsupersymmetric toroidal compactifications are unstable. An analogous mechanism of instability of $AdS^4 \times S^1$ is also studied in [10]; the realization of this type of decay and the stability of the bubbles are discussed in [12]. Horowitz established in [37] that closed string tachyon condensation produces a topology changing transition from black strings to bubble’s of nothing; this provides a dramatic new endpoint to Hawking evaporation. Gibbons and Hartnoll extended the technics of double analytic continuation of black-hole metric used by Witten, to produce generalized bubble of nothing spacetimes [32]. The rotating bubbles are constructed in [27], see also [1]. The previous bubbles of nothing form by quantum tunneling; Brown shows in [18] that bubbles of nothing may also form by thermal fluctuation, or by a mixture of thermal fluctuation and quantum tunneling. The stabilization of the bubbles by a magnetic charge is discussed in [47].

Independently in 1987, S. Hawking investigated in [33] the loss of quantum coherence in Euclidean metrics which have two asymptotically flat regions connected by a wormhole. The role of this wormhole in the quantum gravity is discussed in [34] and [53]. More recently, the Lorentzian version of the Hawking wormhole was considered by Culetu [23]. It turns out that this Lorentzian wormhole is just the submanifold of the Witten space defined by two antipodal points of the Kaluza-Klein dimension S^1 . Therefore, the scalar waves in both spacetimes can be studied in a unified framework.

In contrast with the abundance of this physical litterature, to the best of our knowledge, a rigourous mathematical analysis of the waves propagation on these manifolds is missing. The aim of this paper consists in investigating the Klein-Gordon equation in the Witten spacetime and in the Lorentzian Hawking wormhole. In particular, our work provides a complete mathematical setting

for the physical study of scalar waves made by Bhawal and Vishveshwara [8]. We now describe shortly our geometrical framework. The Witten space-time is constructed as follows (see figure 1). Given $R > 0$, we remove an expanding hole, the “ball of nothing” defined by $|\boldsymbol{\xi}|^2 < \tau^2 + R^2$, from $\mathbb{R}_\tau \times \mathbb{R}_{\boldsymbol{\xi}}^3$, and we endow the resulting manifold \mathcal{M} by a perturbation of the Kaluza-Klein metric

$$ds_{Witten}^2 = d\tau^2 - d\boldsymbol{\xi}^2 - \left(1 - \frac{R^2}{|\boldsymbol{\xi}|^2 - \tau^2}\right) d\psi^2 - \frac{R^2}{|\boldsymbol{\xi}|^2 - \tau^2} \left(\frac{(\tau d\tau + \xi_j d\xi^j)^2}{|\boldsymbol{\xi}|^2 - \tau^2 - R^2}\right).$$

The existence of the fifth dimension ψ has a fundamental consequence: since it shrinks to zero as $|\boldsymbol{\xi}|^2 - \tau^2$ tends to R^2 , the space-time is not singular and has the topology $\mathbb{R}^3 \times S^2$. In particular the set $\mathcal{B} = \{|\boldsymbol{\xi}|^2 - \tau^2 = R^2\}$ is not a boundary but a surface of minimal area isometric with the 2+1-dimensional de Sitter space dS^3

$$\mathcal{B} \equiv dS^3 = \mathbb{R}_\tau \times S_\omega^2, \quad ds_{dS^3}^2 := \frac{R^2}{\tau^2 + R^2} d\tau^2 - (\tau^2 + R^2) d\omega^2.$$

On this submanifold the extra dimension smoothly pinches off, disappearing. From a four-dimensional perspective, this signals the end of spacetime in this region and this bubble has no interior. For this reason \mathcal{B} is termed *bubble of nothing*. Now given two antipodal points $N, S \in S^1$, the submanifold $\{\psi = N, S\}$ is the Lorentzian Hawking wormhole \mathcal{W} . Its equatorial section is depicted below in Figure 2. In suitable coordinates, \mathcal{W} is described by

$$\mathcal{W} = \mathbb{R}_t \times \mathbb{R}_x \times S^2, \quad ds_{\mathcal{W}}^2 = R^2 \cosh^2(x) \left[dt^2 - dx^2 - \cosh^2 t d\Omega_2^2 \right], \quad x \in \mathbb{R}.$$

The throat of the wormhole is the De Sitter submanifold dS^3 located at $x = 0$. As usual, this wormhole is not a vacuum Einstein solution, and the null energy condition is violated. Nevertheless, we establish that it has interesting geometrical properties: its metric is conformally flat and its Ricci scalar is zero; it is weakly traversable, *i.e.* the light rays and the massless fields can across the throat and go to the asymptotically flat infinities, but the time-like geodesics and the massive fields stay near the throat forever.

We now present briefly the structure of the paper and our main results. In section 2, we introduce several sets of coordinates, and we prove that the Witten spacetime is a globally hyperbolic spacetime $\mathbb{R}_t \times \mathbb{R}_{y,z}^2 \times S^2$ where the sections $\Sigma_t = \{t\} \times \mathbb{R}_{y,z}^2 \times S^2$ are Cauchy hypersurfaces and the bubble of nothing is the sub-manifold $y = z = 0$. The angle of the polar coordinates on $\mathbb{R}_{y,z}^2$ is just the Kaluza-Klein dimension ψ . We investigate the causal geodesics in the next section. Any time-like geodesic remains in a bounded domain of $\mathbb{R}_{y,z}^2 \times S^2$. In contrast, the projection on $\mathbb{R}_{y,z}^2$ of the null geodesics that hit \mathcal{B} are whole straight lines. In part 4 we investigate the initial value problem for the Klein-Gordon equation with mass $M \geq 0$

$$\square_g u + M^2 u = 0$$

where \square_g is the D'Alembertian associated with the Witten metric. This equation takes the form

$$(I.1) \quad \left[\partial_t^2 + 2 \tanh t \partial_t - \frac{1}{\cosh^2 t} \Delta_{S^2} + L \right] u = 0,$$

where Δ_{S^2} is the Laplacian on S_ω^2 and the Hamiltonian L is a time-independent differential operator on $\mathbb{R}_{y,z}^2$. We prove that the global Cauchy problem is well posed in the functional framework associated with the energy

$$E(u, t) = |\partial_t u(t)|^2 + \frac{1}{\cosh^2 t} |\nabla_{S^2} u(t)|^2 + |L^{\frac{1}{2}} u(t)|^2$$

where $|\cdot|$ stands for the norm of a suitable L^2 space on $\mathbb{R}_{y,z}^2 \times S_\omega^2$. A fundamental result of this paper is the explicit expression of the solutions established in part 5, that shows the dynamics of the fields is mainly governed by that of the scalar fields in dS^3 : the waves propagating on the Witten

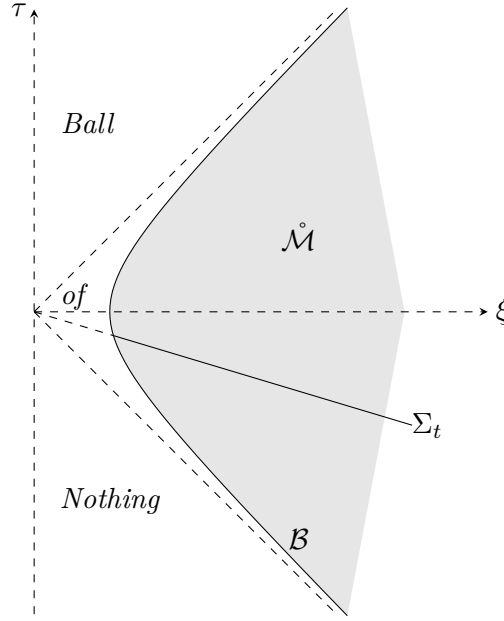


FIGURE 1. The Witten space-time \mathcal{M} is the grey zone. The contracting-expanding ball of nothing $\xi^2 < \tau^2 + R^2$ is deleted. $\mathring{\mathcal{M}}$ is located at $\xi^2 > \tau^2 + R^2$ and each point of $\mathring{\mathcal{M}}$ is $S_\omega^2 \times S_\psi^1$ endowed with the metric $d\omega^2 + \frac{\xi^2 - \tau^2 - R^2}{\xi^2 - \tau^2} d\psi^2$. The *bubble of nothing* \mathcal{B} at $\xi^2 = \tau^2 + R^2$ is not a boundary but just the $2 + 1$ dimensional De Sitter space endowed with the metric $\frac{R^2}{\tau^2 + R^2} d\tau^2 - (\tau^2 + R^2) d\omega^2$. .

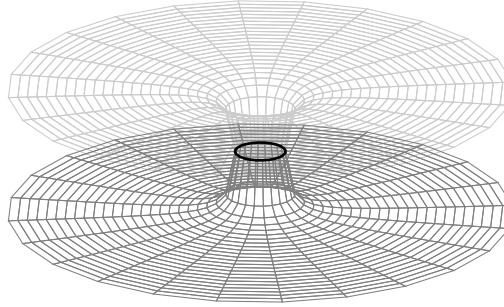


FIGURE 2. Σ_t has the topology of $\mathbb{R}_{y,z}^2 \times S_{\theta,\varphi}^2$. The picture presents its radial-equatorial section $z = 0$, $\theta = \frac{\pi}{2}$. The zone in light (dark) grey is the part $y > 0$, *i.e.* $\psi = 0$, ($y < 0$, *i.e.* $\psi = \pi$). The black ring is the equator of radius $R \cosh t$ at $y = z = 0$ on the bubble of nothing \mathcal{B} . The picture depicts also the section of the Hawking wormhole at time t fixed and $\theta = \frac{\pi}{2}$.

space-time are represented by a Kaluza-Klein tower, *i.e.* a sum of waves on dS^3 . We perform the complete spectral analysis of the self-adjoint operator L . If $\Phi(\lambda, \cdot)$ is the generalized eigenfunction

of L satisfying $L\Phi(\lambda, \cdot) = \lambda\Phi(\lambda, \cdot)$, we prove that

$$u(t, \omega, \cdot) = \int_{\sigma(L)} v_\lambda(t, \omega) \Phi(\lambda, \cdot) d\mu(\lambda)$$

where $d\mu$ is a measure on the spectrum $\sigma(L)$ of L , and v_λ is a solution of the Klein-Gordon equation with mass $\sqrt{\lambda}$ on the De Sitter space dS^3

$$\left[\partial_t^2 + 2 \tanh t \partial_t - \frac{1}{\cosh^2 t} \Delta_{S^2} + \lambda \right] v_\lambda = 0.$$

To describe more precisely $d\mu$, we have to carefully distinguish the massive case from the massless one. Here the mass is the effective mass linked to M and also to the fifth dimension ψ which is the angle of the polar coordinates $(x, \psi) \in (0, \infty) \times S^1$ in the two-plane $\mathbb{R}_{y,z}^2$. We expand the wave u in Fourier series with respect to the Kaluza-Klein dimension and make a separation of variables in the Hamiltonian L :

$$u(t, x, \omega, \psi) = \sum_{n \in \mathbb{Z}} u_n(t, x, \omega) e^{in\psi}, \quad L = \bigoplus_{n \in \mathbb{Z}} L_{M,n}.$$

$L_{M,n}$ is a second-order differential operator on $(0, \infty)_x$, involving the mass $M \geq 0$ and the eigenvalue $n \in \mathbb{Z}$:

$$L_{M,n} = -\frac{1}{\sinh(2x)} \partial_x (\sinh(2x) \partial_x) + (M^2 + n^2) \cosh^2 x + n^2 \coth^2 x.$$

$n = 0$ corresponds to the ordinary matter (u does not depend on the fifth dimension), while the Kaluza-Klein particles are associated with $n \neq 0$ and are always massive (see [5]). We say that the field u_n is massive if its effective mass $\sqrt{M^2 + n^2}$ is not zero. In this case, the potential $(M^2 + n^2) \cosh^2 x$ is confining hence the spectrum of $L_{M,n}$ is discrete and included in $(1, \infty)$. In contrast, in the massless case $M = n = 0$, the spectrum of $L_{0,0}$ is absolutely continuous and equal to $[1, \infty)$. These properties allow to investigate in part 6 the asymptotic behaviours of the fields. Taking account of the exponential damping due to the fast expansion, we consider the profile v of the scalar field u ,

$$v(t, \cdot) := (\cosh t) u(t, \cdot)$$

that are solutions of

$$\left[\partial_t^2 - \frac{1}{\cosh^2 t} \Delta_{S^2} + L - 1 \right] v = 0.$$

We compare v with the solutions v_\sharp of

$$\left[\partial_t^2 + L - 1 \right] v_\sharp = 0,$$

that are quasi-periodic or dispersive and we prove that $v(t) \sim v_{in(out)}(t)$ as $t \rightarrow -(+)\infty$. The main result of this section assures that if the effective mass is not zero, v is asymptotically quasi-periodic as $|t|$ tends to infinity, and if the effective mass is zero, $M = 0$, $u = u_0$, then v is dispersive. The seventh part is devoted to the presentation of the geometrical properties of the Lorentzian Hawking wormhole that are not known in the litterature. Its Ricci scalar is zero and this wormhole is weakly traversable, *i.e.* the light ray can cross the throat and go from a sheet to the other sheet but the time-like geodesics stay in the vicinity of the contracting-expanding throat. In part 8 we study the Klein-Gordon equation in the Hawking wormhole. Its form is (I.1) again where now L is a differential operator on \mathbb{R}_x . We get similar results: the wormhole is globally hyperbolic and the global Cauchy problem is well posed in the finite energy spaces. The profile v of the field is asymptotically quasi-periodic if $M > 0$, but if $M = 0$, v is asymptotically free, $v(t, x, \omega) \sim v_{in(out)}^+(x + t, \omega) + v_{in(out)}^-(x - t, \omega)$, $t \rightarrow -(+)\infty$. Therefore the Lorentzian Hawking wormhole is traversable by the fields iff the mass is zero. All the previous results are used in the last part to establish the most important result of this work: the existence of the classical and quantum scattering operators $S : v_{in} \mapsto v_{out}$ for the Witten spacetime and the Hawking wormhole. We prove that these operators are isomorphisms on the one-particle Hilbert spaces and they are

unitarily implementable in the Fock-Cook quantization. The key point is that there is no mixing between the positive and the negative frequencies. As a striking consequence, the quantized scattering operator leaves invariant the Fock vacuum, *i.e.* there is no creation of particles despite the time-dependence of the Witten and Hawking metrics.

II. THE WITTEN SPACE-TIME

In this part we describe the Witten spacetime. In particular we present several choices of coordinates that allow to rigorously prove the statement of Witten in [54], that this spacetime is a smooth manifold without boundary. The main result of this section is the theorem of global hyperbolicity. Recall that Witten obtained its model by considering the 5-dimensional Schwarzschild metric

$$ds^2 = \left(1 - \frac{R^2}{\rho^2}\right) dT^2 - \left(1 - \frac{R^2}{\rho^2}\right)^{-1} d\rho^2 - \rho^2 (d\Theta^2 + \sin^2 \Theta d\Omega_2^2), \quad \rho > R,$$

where $R > 0$ is given and $d\Omega_2^2$ is the line element of the two dimensional sphere S^2 . We get another vacuum solution of the 5D Einstein equations by the double analytic continuation

$$T = i\psi, \quad \Theta = \frac{\pi}{2} + it.$$

To avoid a conic singularity at $\rho = R$ we require ψ to be 2π periodic, hence we denote $\psi \in [0, 2\pi)$ or $\Omega_1 \in S^1$ the Kaluza-Klein dimension. We shall see that $\rho = R$ does not locate a boundary, but a surface of minimal area, the *bubble of nothing*, that is just the 3-dimensional De Sitter space-time dS^3 :

$$ds_{dS^3}^2 := R^2 [dt^2 - \cosh^2 t d\Omega_2^2].$$

At this step, we have constructed the exterior of the Witten bubble of nothing, that is the 5-dimensional space-time

$$(II.1) \quad \mathring{\mathcal{M}} = \mathbb{R}_t \times]R, \infty[_\rho \times S_{\Omega_2}^2 \times S_{\Omega_1}^1,$$

$$(II.2) \quad ds_{Witten}^2 = g_{\mu\nu} dx^\mu dx^\nu := \rho^2 dt^2 - \left(1 - \frac{R^2}{\rho^2}\right)^{-1} d\rho^2 - \rho^2 \cosh^2 t d\Omega_2^2 - \left(1 - \frac{R^2}{\rho^2}\right) d\Omega_1^2$$

where S^d is the d -dimensional unit sphere and $d\Omega_d^2$ its usual metric. To study this manifold and to investigate what happens if $\rho = R$, we use the Rindler coordinates associated with the Minkowski metric $d\tau^2 - d\xi^2 = \rho^2 dt^2 - d\rho^2$ on the Rindler wedge $\xi > |\tau|$:

$$(II.3) \quad \tau := \rho \sinh t, \quad \xi := \rho \cosh t,$$

hence the Witten metric becomes

$$(II.4) \quad g_{\mu\nu} dx^\mu dx^\nu = d\tau^2 - d\xi^2 - \xi^2 d\Omega_2^2 - d\Omega_1^2 + \frac{R^2}{\xi^2 - \tau^2} \left\{ d\Omega_1^2 - \frac{(\tau d\tau - \xi d\xi)^2}{\xi^2 - \tau^2 - R^2} \right\},$$

on the set $\tau \in \mathbb{R}$, $\xi > \sqrt{\tau^2 + R^2}$, $\Omega_d \in S^d$. If we think ξ as the radial coordinate $\xi := |\boldsymbol{\xi}|$, $\boldsymbol{\xi} = \xi \Omega_2$, of the Minkowski space-time $\mathbb{R}_\tau \times \mathbb{R}_{\boldsymbol{\xi}}^3$, $\mathring{\mathcal{M}}$ looks like a distorted Kaluza-Klein space-time $\mathbb{R}_\tau \times \mathbb{R}_{\boldsymbol{\xi}}^3 \times S^1$ where the contracting-expanding “ball of nothing” $|\boldsymbol{\xi}|^2 \leq \tau^2 + R^2$ has been deleted (Figure 1).

At first glance, the “bubble of nothing” $|\boldsymbol{\xi}|^2 = \tau^2 + R^2$ could define a boundary. In fact this is not the case and $\mathring{\mathcal{M}}$ can be extended into a Lorentzian manifold \mathcal{M} without boundary that is

globally hyperbolic. To establish this fundamental property, it will be convenient to introduce a new radial coordinate

$$(II.5) \quad r := R^{-1} \sqrt{\rho^2 - R^2},$$

for which the metric becomes

$$(II.6) \quad ds_{Witten}^2 = R^2 \left\{ (r^2 + 1) dt^2 - g_{ij}(t) dx^i dx^j \right\},$$

$$(II.7) \quad g_{ij}(t) dx^i dx^j := dr^2 + (r^2 + 1) \cosh^2 t (d\theta^2 + \sin^2 \theta d\varphi^2) + \frac{r^2}{r^2 + 1} d\psi^2$$

with $t \in \mathbb{R}$, $r \in]0, \infty[$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$, $\psi \in [0, 2\pi]$, and $i, j \in \{1, 2, 3\}$. Without loss of generality we assume in the sequel $R = 1$. We note that $\mathring{\mathcal{M}} = \mathbb{R}_t \times \mathring{\Sigma}$, $\mathring{\Sigma} :=]0, \infty[\times S^2 \times S^1$, is a Lorentzian manifold with $r^2 + 1$ as a lapse function, and for each $t \in \mathbb{R}$, the slice $\mathring{\Sigma}_t := \{t\} \times \mathring{\Sigma}$ is a Riemannian manifold endowed with the metric $g_{ij}(t)$ given by (II.7). The crucial point is that the completion Σ_t of $\mathring{\Sigma}_t$ has no boundary. To see that, we put

$$(II.8) \quad y := \frac{r e^{\sqrt{r^2+1}}}{1 + \sqrt{r^2+1}} \cos \psi, \quad z := \frac{r e^{\sqrt{r^2+1}}}{1 + \sqrt{r^2+1}} \sin \psi,$$

and we get

$$(II.9) \quad g_{ij}(t) dx^i dx^j = \frac{(1 + \sqrt{r^2+1})^2}{r^2 + 1} e^{-2\sqrt{r^2+1}} (dy^2 + dz^2) + (r^2 + 1) \cosh^2 t d\Omega_2^2$$

where $(y, z) \in \mathbb{R}^2$ and r^2 is a smooth function of (y, z) implicitly defined by

$$(II.10) \quad r^2 e^{2\sqrt{r^2+1}} (1 + \sqrt{r^2+1})^{-2} = y^2 + z^2.$$

In fact, this equation is easily solved in terms of the generalized Lambert function $W^{(+2)}_{-2}(x)$ introduced in [6], that is solution of the transcendental equation

$$(II.11) \quad \frac{W(x) - 2}{W(x) + 2} e^{W(x)} = x,$$

and we have:

$$(II.12) \quad \sqrt{r^2 + 1} = \frac{1}{2} W^{(+2)}_{-2}(y^2 + z^2).$$

In particular, this function is real analytic and near the origin we have

$$(II.13) \quad \sqrt{r^2 + 1} = 1 - 2 \sum_{n=1}^{\infty} \frac{L'_n(4n)}{n e^{2n}} (y^2 + z^2)^n$$

where L'_n is the derivative of the n th Laguerre polynomial. We deduce that

$$(II.14) \quad \Sigma_t := \mathring{\Sigma}_t \cup (\{t\} \times \{(y, z) = (0, 0)\})$$

is a C^∞ Riemannian manifold that is complete, and that $r = 0$ (or $\rho = R$, or $\xi = \sqrt{\tau^2 + R^2}$) is not associated with a boundary or a horizon: it is just a pseudo-singularity of coordinate, exactly like the origin in spherical coordinate and Σ_0 , which is asymptotic to $\mathbb{R}^3 \times S^1$ with the flat metric as $r \rightarrow \infty$, has the topology of $\mathbb{R}_{y,z}^2 \times S^2$ (see Figure 2). Moreover $r = 0$, *i.e.* the submanifold $\{0_{\mathbb{R}_{y,z}^2}\} \times S^2$, is a surface of minimal area and the Witten bubble of nothing $(\mathcal{B}, g_{\alpha\beta})$ defined by

$$(II.15) \quad \mathcal{B} := \mathbb{R}_t \times \{0_{\mathbb{R}_{y,z}^2}\} \times S^2, \quad g_{\alpha\beta} dx^\alpha dx^\beta = dt^2 - \cosh^2 t d\Omega_2^2 = \frac{1}{\tau^2 + 1} d\tau^2 - (\tau^2 + 1) d\Omega_2^2$$

is just a $(1 + 2)$ -dimensional De Sitter spacetime. Finally the Witten spacetime defined by

$$(II.16) \quad \mathcal{M} := \mathring{\mathcal{M}} \cup \mathcal{B} = \mathbb{R}_t \times \Sigma, \quad \Sigma := \mathbb{R}_{y,z}^2 \times S^2,$$

$$(II.17) \quad ds_{Witten}^2 = (r^2 + 1)dt^2 - \frac{(1 + \sqrt{r^2 + 1})^2}{r^2 + 1} e^{-2\sqrt{r^2 + 1}} (dy^2 + dz^2) - (r^2 + 1) \cosh^2 t d\Omega_2^2,$$

where r^2 is given by (II.12), is a C^∞ Lorentzian manifold without boundary.

Proposition II.1. *The Witten space-time (\mathcal{M}, g) is globally hyperbolic and Σ_t is a Cauchy hypersurface.*

Proof. Since the coefficients of the metric do not satisfy the assumptions of the usual criteria of global hyperbolicity (Theorem 2.1 of [21], Theorem 6.1 of [46]), we make a direct demonstration. We consider an inextendible causal curve $\gamma : \lambda \in (a, b) \mapsto \gamma(\lambda) = (t(\lambda), y(\lambda), z(\lambda), \Omega_2(\lambda)) \in \mathbb{R}^3 \times S^2$. We have to prove that for any t , Σ_t is cut once by γ . Since \mathcal{M} is globally time oriented by $\frac{\partial}{\partial t}$ and g is invariant by time-reversing, it is sufficient to consider a future directed causal curve passing through $(t_0, y_0, z_0, \Omega_{2,0})$ at some λ_0 and we prove that γ cuts once Σ_t when $t > t_0$. Since $\dot{t}(\lambda) > 0$, we have to prove that $t(\lambda) \rightarrow +\infty$ as $\lambda \rightarrow b$. We assume that $t(\lambda) \rightarrow t_* < \infty$ as $\lambda \rightarrow b$. Since γ is causal future-directed, we have

$$\dot{t} \geq \left[\left(\frac{1 + \sqrt{r^2 + 1}}{r^2 + 1} e^{-\sqrt{r^2 + 1}} \right)^2 (\dot{y}^2 + \dot{z}^2) + \cosh^2 t |\dot{\Omega}_2|^2 \right]^{\frac{1}{2}}.$$

(II.8) assures that $y^2 + z^2 \sim e^{2r}$ at the infinity and $y^2 + z^2 \sim r^2$ near 0. We deduce that for all $\lambda_1 \in (\lambda_0, b)$ we have

$$t_* - t_0 \gtrsim \int_{\lambda_0}^{\lambda_1} \sqrt{\frac{\dot{y}^2 + \dot{z}^2}{1 + y^2 + z^2}} d\lambda \gtrsim \int_{\lambda_0}^{\lambda_1} \left| \frac{d}{d\lambda} \ln(1 + y^2 + z^2) \right| d\lambda \geq \left| \ln \left(\frac{y^2(\lambda_1) + z^2(\lambda_1) + 1}{y_0^2 + z_0^2 + 1} \right) \right|$$

hence $y(\lambda)$ and $z(\lambda)$ are bounded as $\lambda \rightarrow b$ and we have on $[\lambda_0, b)$:

$$\dot{t} \gtrsim \sqrt{\dot{y}^2 + \dot{z}^2 + |\dot{\Omega}_2|^2}.$$

We conclude that $\gamma(t)$ tends to some point $(t_*, y_*, z_*, \Omega_{2,*})$ in $\mathbb{R}^3 \times S^2$ as $\lambda \rightarrow b$ and therefore γ is not inextendible. This contradiction achieves the proof.

Q.E.D.

We end this part by some remarks on the asymptotics of the Witten metric. As $\rho \rightarrow \infty$, we have

$$(II.18) \quad ds_{Witten}^2 \sim \rho^2 dt^2 - d\rho^2 - \rho^2 \cosh^2 t d\Omega_2^2 - d\Omega_1^2 = d\tau^2 - d\xi^2 - \xi^2 d\Omega_2^2 - d\Omega_1^2,$$

therefore the Witten space-time is asymptotically described at the spacelike infinity, (i) in the (t, ρ) coordinates, by the cartesian product of the 4D Rindler space-time and the circle S^1 , (ii) in the (τ, ξ) coordinates by the Kaluza-Klein space-time. In contrast, at the timelike infinity, the Witten metric behaves like the De Sitter metric: if we introduce $x := \operatorname{arcsinh} r \in [0, \infty)$, we get

$$(II.19) \quad ds_{Witten}^2 = \cosh^2 x \left[dt^2 - dx^2 - \cosh^2 t d\Omega_2^2 - \frac{\sinh^2 x}{\cosh^4 x} d\Omega_1^2 \right].$$

Finally we introduce new coordinates that allow to see that the Witten metric is conformally equivalent with simpler metrics. We put

$$(II.20) \quad \sigma := \frac{\rho + \sqrt{\rho^2 - 1}}{2} \in \left[\frac{1}{2}, \infty \right) \quad (R = 1).$$

Then

$$(II.21) \quad ds_{Witten}^2 = \left(1 + \frac{1}{4\sigma^2} \right)^2 \left\{ \sigma^2 dt^2 - d\sigma^2 - \sigma^2 \cosh^2 t d\Omega_2^2 - 16\sigma^4 \frac{(4\sigma^2 - 1)^2}{(4\sigma^2 + 1)^4} d\Omega_1^2 \right\},$$

hence we can see that the sub-manifold $\Omega_1 = Cst.$ is conformally flat. Since (t, σ, Ω_2) are Rindler-type coordinates again, we can introduce

$$(II.22) \quad T := \sigma \sinh t, \quad \Sigma := \sigma \cosh t,$$

for which the Witten manifold is defined by $\Sigma^2 - T^2 \geq \frac{1}{4}$, $\Sigma \geq \frac{1}{2}$, and

$$(II.23) \quad ds_{Witten}^2 = \left(1 + \frac{1}{4(\Sigma^2 - T^2)}\right)^2 \left\{ dT^2 - d\Sigma^2 - \Sigma^2 d\Omega_2^2 - 16(\Sigma^2 - T^2)^2 \frac{[4(\Sigma^2 - T^2) - 1]^2}{[4(\Sigma^2 - T^2) + 1]^4} d\Omega_1^2 \right\}.$$

We deduce that when $\Sigma \rightarrow \infty$, $T = \pm \Sigma + T_0$ we have

$$ds_{Witten}^2 \sim dT^2 - d\Sigma^2 - \Sigma^2 d\Omega_2^2 - d\Omega_1^2.$$

Therefore the Witten spacetime looks like the Kaluza-Klein spacetime at the future/past null infinity.

III. CAUSAL GEODESICS IN THE WITTEN SPACETIME

The geodesic motion in the Witten spacetime has been discussed by Brill and Matlin in [15] (see also [1]). In this part we complete this study by precisely analysing the causal geodesics that hit the bubble of nothing. In Schwarzschild coordinates a geodesic γ is expressed as

$$\gamma : \lambda \in \mathbb{R} \mapsto (t(\lambda), \rho(\lambda), \omega(\lambda), \psi(\lambda)) \in \mathbb{R} \times [R, \infty[\times S^2 \times S^1,$$

and with the (t, y, z, ω) coordinates we write

$$\gamma : \lambda \in \mathbb{R} \mapsto (t(\lambda), y(\lambda), z(\lambda), \omega(\lambda)) \in \mathbb{R} \times \mathbb{R}^2 \times S^2.$$

In the computations, S^1 is identified with $\mathbb{R}/2\pi\mathbb{Z}$ and ψ is a real valued function, and S^2 is described by $(\theta, \varphi) \in [0, \pi] \times \mathbb{R}^2/2\pi\mathbb{Z}$. Outside the bubble of nothing $\rho = R$, we use the Schwarzschild type coordinates $(t, \rho, \theta, \varphi, \psi)$, for which the geodesic equations are

$$(III.1) \quad \ddot{t} + \frac{2}{\rho} \dot{t} \dot{\rho} + \sinh t \cosh t (\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2) = 0,$$

$$(III.2) \quad \ddot{\rho} - \frac{R^2}{\rho^3} \left(1 - \frac{R^2}{\rho^2}\right)^{-1} \dot{\rho}^2 + \rho \left(1 - \frac{R^2}{\rho^2}\right) \ddot{t} - \rho \left(1 - \frac{R^2}{\rho^2}\right) \cosh^2 t (\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2) - \frac{R^2}{\rho^3} \left(1 - \frac{R^2}{\rho^2}\right) \dot{\psi}^2 = 0,$$

$$(III.3) \quad \ddot{\theta} + 2 \frac{\sinh t}{\cosh t} \dot{t} \dot{\theta} + \frac{2}{\rho} \dot{\theta} \dot{\varphi} - \sin \theta \cos \theta \dot{\varphi}^2 = 0,$$

$$(III.4) \quad \ddot{\varphi} + 2 \frac{\sinh t}{\cosh t} \dot{t} \dot{\varphi} + \frac{2}{\rho} \dot{\rho} \dot{\varphi} + 2 \frac{\cos \theta}{\sin \theta} \dot{\theta} \dot{\varphi} = 0,$$

$$(III.5) \quad \ddot{\psi} + 2 \frac{R^2}{\rho^3} \left(1 - \frac{R^2}{\rho^2}\right)^{-1} \dot{\rho} \dot{\psi} = 0.$$

We know that

$$(III.6) \quad E := g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

is a constant along a geodesic $x^\mu(\lambda)$ with $E > 0$ for the time-like geodesics and $E = 0$ for the null geodesics. We note that for a causal geodesic \dot{t} never is zero, $t(\mathbb{R}) = \mathbb{R}$ by the global hyperbolicity of \mathcal{M} , and we have $\cosh^2 t \dot{\varphi}^2 \leq \dot{t}^2$. We deduce that for a future directed causal geodesic we have

$$\left| \frac{d\varphi}{dt} \right| \leq \frac{1}{\cosh t}$$

and we get that

$$(III.7) \quad \exists \varphi_{\pm}, \quad \varphi(\lambda) \rightarrow \varphi_{\pm}, \quad \lambda \rightarrow \pm\infty.$$

Therefore the causal future of a point $(t_0, \rho_0, \theta_0, \varphi_0, \psi_0)$ is a set of point $(t, \rho, \theta, \varphi, \psi)$ satisfying

$$(III.8) \quad t_0 \leq t, \quad |\varphi - \varphi_0| \leq 2 \left[\arctan(e^t) - \arctan(e^{t_0}) \right].$$

This inequality is optimal since we can easily check that given $K \neq 0$, $\rho_0 > R$, $\psi_0 \in S^1$, the path defined by $t = \operatorname{arcsinh}(K\lambda)$, $\theta = \pi/2$, $\rho = \rho_0$, $\varphi = \arctan(K\lambda)$, $\psi = \psi_0$ is a null geodesic satisfying $d\varphi/dt = 1/\cosh t$. We conclude there exists a horizon associated with any observer, similar to the cosmological horizon in the De Sitter universe. In particular, two points $(t_0, \rho_j, \theta_j, \varphi_j, \psi_j)$, $\varphi_j \in [0, 2\pi[$, $j = 1, 2$, are causally disconnected in the future if

$$\min(|\varphi_1 - \varphi_2|, 2\pi - |\varphi_1 - \varphi_2|) > 2\pi - 4 \arctan(e^{t_0}).$$

In constrast, there exists time-like geodesics that are rotating along the whole fifth dimension S^1 . If we choose $E > 0$, $\rho_0 \in]R, \sqrt{2}R[$, $\omega_0 \in S^2$, the path $\gamma(\lambda) = \left(t = \frac{R\sqrt{E}}{\rho_0\sqrt{2R^2 - \rho_0^2}}\lambda, \rho_0, \omega_0, \psi = \frac{\rho_0\sqrt{E}}{\sqrt{2R^2 - \rho_0^2}}\lambda \right)$ is a geodesic of this type. For $\rho = \sqrt{2}R$, we have a null geodesic $\gamma(\lambda) = (t = \lambda, \rho = \sqrt{2}R, \omega_0, \psi = 2R\lambda)$.

We note that given a geodesic $\gamma(\lambda) = (t(\lambda), \rho(\lambda), \theta(\lambda), \varphi(\lambda), \psi(\lambda))$, we can choose a frame such that $\theta(0) = \frac{\pi}{2}$, $\dot{\theta}(0) = 0$. Hence γ is included in the hypersurface $\theta = \frac{\pi}{2}$ for any value of the proper time λ . For simplicity, we assume in the sequel that $\theta(\lambda)$ is constant and equal to $\frac{\pi}{2}$ and we compute several conserved quantities $\xi_\mu \dot{x}^\mu$ associated with several Killing vectors ξ^μ . Since $\frac{\partial}{\partial \varphi}$ and $\frac{\partial}{\partial \psi}$ are Killing vectors, we have two constants of the motion:

$$(III.9) \quad K_\varphi := (\rho^2 \cosh^2 t) \dot{\varphi}, \quad K_\psi := \left(1 - \frac{R^2}{\rho^2}\right) \dot{\psi}.$$

We deduce that φ and ψ are monotone functions and if $K_\psi \neq 0$ we have $\psi(\mathbb{R}) = S^1$. Associated with the boosts $\cos \varphi \frac{\partial}{\partial t} - \tanh t \sin \varphi \frac{\partial}{\partial \varphi}$, $\sin \varphi \frac{\partial}{\partial t} + \tanh t \cos \varphi \frac{\partial}{\partial \varphi}$, we have two other conserved quantities:

$$(III.10) \quad K'_t := (\rho^2 \cos \varphi) \dot{t} + (\rho^2 \sinh t \cosh t \sin \varphi) \dot{\varphi}, \quad K''_t := (\rho^2 \sin \varphi) \dot{t} - (\rho^2 \sinh t \cosh t \cos \varphi) \dot{\varphi}.$$

By using the previous results we obtain

$$(III.11) \quad \dot{\rho}^2 + K_\psi^2 + \left(1 - \frac{R^2}{\rho^2}\right) \left[\frac{K_\varphi^2 - K_t'^2 - K_t''^2}{\rho^2} + E \right] = 0.$$

For a time-like geodesic, $K_t'^2 + K_t''^2 - K_\varphi^2$ is a positive constant and we have

$$(III.12) \quad R \leq \rho \leq R^* := \left(\frac{K_t'^2 + K_t''^2 - K_\varphi^2}{E} \right)^{\frac{1}{2}} = \rho(0) \left(1 + \frac{\left(1 - \frac{R^2}{\rho^2(0)}\right)^{-1} \dot{\rho}^2(0) + \left(1 - \frac{R^2}{\rho^2(0)}\right) \dot{\psi}^2(0)}{E} \right)^{\frac{1}{2}}.$$

To investigate the geodesics crossing the singularity of coordinate $\rho = R$, and to avoid the trouble in the equations (III.2) and (III.5), we use the $(t, y, z, \omega) \in \mathbb{R} \times \mathbb{R}^2 \times S^2$ coordinates, for which the Witten metric is (we take in the sequel $R = 1$):

$$g_{\mu\nu} dx^\mu dx^\nu = \rho^2 dt^2 - \frac{(1 + \rho)^2}{\rho^2} e^{-2\rho} (dy^2 + dz^2) - \rho^2 \cosh^2 t d\Omega_2^2,$$

where ρ is the C^∞ function of (y, z) given by (II.12):

$$(III.13) \quad \rho = \frac{1}{2}W \left(\begin{smallmatrix} +2 \\ -2 \end{smallmatrix}, y^2 + z^2 \right).$$

The angular momentum K_ψ introduced in (III.9) can be written as

$$(III.14) \quad K_\psi = F(y, z)(y\dot{z} - \dot{y}z), \quad F(y, z) := \frac{\rho + 1}{\rho^2} \frac{\rho - 1}{y^2 + z^2}.$$

Thanks to (II.13), we can see that $F \in C^\infty(\mathbb{R}^2)$ and F is strictly positive (in particular $F(0, 0) = 4e^{-2}$). We deduce that when a geodesic crosses $y = z = 0$ at some proper time λ , then $y\dot{z} - \dot{y}z = 0$ at each time, *i.e.* its projection on the two-plane $\mathbb{R}_y \times \mathbb{R}_z$ is included in a straight line crossing the origin. Without loss of generality, we consider the case $z = 0$ and $\theta = \pi/2$, so, taking into account the conservation of E , the geodesic equation for y becomes:

$$(III.15) \quad \ddot{y} - \frac{\rho}{\rho + 1} y \dot{y}^2 W' \left(\begin{smallmatrix} +2 \\ -2 \end{smallmatrix}, y^2 \right) + E \frac{y}{\rho} \left(\frac{\rho + 1}{\rho + 2} \right)^2 = 0.$$

We note that $W' = \left(1 + \frac{2}{W}\right)^2 e^{-W} > 0$. First we consider the case of a null geodesic, *i.e.* $E = 0$. The previous equation has the form

$$\ddot{y} = A(\lambda)y, \quad 0 \leq A.$$

We deduce that if $y(0) = 0$ and $\dot{y}(0) \neq 0$, then $|\dot{y}|$ is never zero and is a strictly monotone function. Hence y is also strictly monotone and $y(\mathbb{R}) = \mathbb{R}$. We conclude that the projection on the (y, z) -plane of a null geodesic crossing the bubble of nothing $\rho = R$, is a whole straight line crossing the origin. Now we consider a time-like geodesic, *i.e.* $E > 0$, with $y(0) = 0$, $\dot{y}(0) > 0$, $z = 0$, and we show that y is λ -periodic. R^* being defined by (III.12), we introduce

$$(III.16) \quad \lambda^* := \sqrt{E} \int_1^{R^*} \left(1 - \frac{1}{\varrho^2}\right)^{-\frac{1}{2}} \left(\frac{R^*}{\varrho^2} - 1\right)^{-\frac{1}{2}} d\varrho.$$

(III.11) becomes

$$(III.17) \quad \dot{\rho}^2 = E \left(1 - \frac{1}{\rho^2}\right) \left(\frac{R^*}{\rho^2} - 1\right),$$

and in term of the y unknown:

$$(III.18) \quad \dot{y}^2 = E \frac{F(y, 0)}{W'^2 \left(\begin{smallmatrix} +2 \\ -2 \end{smallmatrix}, y^2 \right)} \left(\frac{R^*}{\rho^2} - 1\right),$$

where F is given by (III.14). We note that $\dot{\rho}(\lambda) = 0$ iff $\rho(\lambda) = 1$ or $\rho(\lambda) = R^*$, and $\dot{y}(\lambda) = 0$ iff $\rho(\lambda) = R^*$. We deduce from (III.17) and (III.18) that ρ and y are increasing functions for $\lambda \in (0, \lambda^*)$, and $\rho(\lambda^*) = R^*$, $y(\lambda^*) = y^*$. The geodesic equation (III.15) implies $\ddot{y}(\lambda^*) < 0$, hence ρ and y are decreasing functions for $\lambda \in (\lambda^*, 2\lambda^*)$, and $\rho(2\lambda^*) = 1$, $y(2\lambda^*) = 0$, $\dot{y}(2\lambda^*) = -\dot{y}(0)$. By the same argument, we have $\rho(3\lambda^*) = R^*$, $y(3\lambda^*) = -y^*$, $\ddot{y}(3\lambda^*) > 0$, and finally $y(4\lambda^*) = 0$, $\dot{y}(4\lambda^*) = \dot{y}(0)$. We conclude that the (y, z) -coordinates of the time like geodesics crossing the origin are periodic functions of λ .

To end this study, we consider the causal geodesics that stay at $y = z = 0$. It is easy to check that given $E > 0$, $\omega_0 \in S^2$, the path $\gamma(\lambda) = (t = \sqrt{E}\lambda, y = z = 0, \omega_0)$ is a time-like geodesic, and given $K_\varphi \neq 0$, the path $\gamma(\lambda) = (t = \text{arcsinh}(K_\varphi \lambda), y = z = 0, \theta = \pi/2, \varphi = \arctan(K_\varphi \lambda))$ is a null geodesic. We summarize the main properties of the causal geodesics.

Proposition III.1. *Let γ a causal geodesic in \mathcal{M} . Then $\dot{t}(\lambda) \neq 0$, $t(\mathbb{R}) = \mathbb{R}$. $\omega(\mathbb{R})$ is included in a half of a great circle of S^2 and there exists $\lim_{\lambda \rightarrow \pm\infty} \omega(\lambda)$. For instance, $t = \text{arcsinh}(\lambda)$, $\rho = \rho_0 > R$, $\theta = \pi/2$, $\varphi = \arctan(\lambda)$, $\psi = \psi_0$, defines a null geodesic.*

Two points $(t_0, y_k, z_k; \omega_k) \in \mathbb{R}^3 \times S^2$, $k = 1, 2$, are causally disconnected in the future if the distance between ω_1 and ω_2 on S^2 is larger than $2\pi - 4 \arctan(e^{t_0})$.

If γ is time-like, then ρ is a bounded function of λ .

If γ does not hit the bubble of nothing $\rho = R$, then $\psi(\mathbb{R}) = S^1$. For instance, given $\rho_0 \in]R, \sqrt{2}R[$, $t = \frac{R}{\rho_0 \sqrt{2R^2 - \rho_0^2}} \lambda$, $\rho(\lambda) = \rho_0$, $\omega(\lambda) = \omega_0 \in S^2$, $\psi(\lambda) = \frac{\rho_0}{\sqrt{2R^2 - \rho_0^2}} \lambda$ defines a time-like geodesic, and the path $(t = \lambda, \rho = \sqrt{2}R, \omega_0, \psi = 2R\lambda)$ is a null geodesic.

If γ hits the bubble of nothing but does not stay on it, then $\psi(\mathbb{R})$ is a pair of two antipodal points. In this case $(y(\lambda), z(\lambda))$ delineates, either a whole straight line crossing $(0, 0)$ if γ is null, or a straight segment crossing the origin in its middle if γ is time-like, and then y and z are λ -periodic.

There exists causal geodesics that stay on the bubble of nothing: given $\omega_0 \in S^2$, $t = \lambda, y = z = 0, \omega = \omega_0$ defines a time-like geodesic, and $t = \text{arcsinh}(\lambda), y = z = 0, \theta = \pi/2, \varphi = \arctan(\lambda)$ is a null geodesic.

IV. KLEIN-GORDON FIELDS ON THE WITTEN SPACE-TIME

The global hyperbolicity of \mathcal{M} assures that the global Cauchy problem of the linear relativistic wave equations with data specified on Σ_{t_0} is well posed in C_0^∞ , C^∞ and in the space of distributions (Theorems of Leray [41], see also [22]). In this section, we investigate the scalar waves in a Hilbertian framework associated with the energy that is suitable to develop a scattering theory. Given $M \geq 0$ and $t_0 \in \mathbb{R}$, we consider the Cauchy problem associated to the Klein-Gordon equation:

$$(IV.1) \quad \square_g u + M^2 u = 0 \text{ in } \mathcal{M},$$

$$(IV.2) \quad u = f, \quad \partial_t u = g \text{ on } \Sigma_{t_0},$$

where $\square_g := \frac{1}{\sqrt{|\det(g)|}} \partial_\mu (\sqrt{|\det(g)|} g^{\mu\nu} \partial_\nu)$. In (t, y, z, Ω_2) coordinates, the equation has the form:

$$(IV.3) \quad \frac{1}{\cosh^2 t} \partial_t (\cosh^2 t \partial_t u) - \frac{e^{2\sqrt{r^2+1}} \sqrt{r^2+1}}{(1 + \sqrt{r^2+1})^2} \left[\partial_y \left((r^2+1)^{\frac{3}{2}} \partial_y u \right) + \partial_z \left((r^2+1)^{\frac{3}{2}} \partial_z u \right) \right] - \frac{1}{\cosh^2 t} \Delta_{S^2} u + M^2 (r^2+1) u = 0,$$

where r^2 is given by (II.10) or (II.12). To choose the functional spaces, we remark that if u is a smooth solution, compactly supported at each time, for instance $u \in C^2(\mathbb{R}_t; C_0^2(\Sigma))$, we have the following energy estimate:

$$(IV.4) \quad \frac{d}{dt} E(u, t) = -2 \tanh t \int_{\Sigma_t} \left[2 |\partial_t u|^2 + \frac{1}{\cosh^2 t} |\nabla_{S^2} u|^2 \right] d\mu$$

where

$$(IV.5) \quad E(u, t) := \int_{\Sigma_t} \left[|\partial_t u|^2 + \frac{(r^2+1)^2}{(1 + \sqrt{r^2+1})^2} e^{2\sqrt{r^2+1}} |\nabla_{\mathbb{R}^2} u|^2 + \frac{1}{\cosh^2 t} |\nabla_{S^2} u|^2 + M^2 (r^2+1) |u|^2 \right] d\mu$$

here $\nabla_{\mathbb{R}^2} u := (\partial_y u, \partial_z u)$ and the measure μ on $\Sigma = \mathbb{R}^2 \times S^2$ is given by

$$(IV.6) \quad d\mu := d\nu \otimes d\Omega_2, \quad d\nu := \frac{(1 + \sqrt{r^2+1})^2}{\sqrt{r^2+1}} e^{-2\sqrt{r^2+1}} dy dz = r dr d\psi.$$

We introduce the norms

$$(IV.7) \quad \|u\|_{Y^1}^2 := \int_{\Sigma} \left[\frac{(r^2 + 1)^2}{(1 + \sqrt{r^2 + 1})^2} e^{2\sqrt{r^2 + 1}} |\nabla_{\mathbb{R}^2} u|^2 + |\nabla_{S^2} u|^2 + (r^2 + 1) |u|^2 \right] d\mu$$

and the Hilbert spaces

$$(IV.8) \quad X^0 := L^2(\Sigma, d\mu), \quad Y^1 := \left\{ u \in X^0, \quad \|u\|_{Y^1} < \infty \right\}.$$

It is clear that $C_0^\infty(\Sigma)$ is dense in X^0 . We remark that the constant functions do not belong to Y^1 and

$$(IV.9) \quad \|u\|_{W^1}^2 := \int_{\Sigma} \left[\frac{(r^2 + 1)^2}{(1 + \sqrt{r^2 + 1})^2} e^{2\sqrt{r^2 + 1}} |\nabla_{\mathbb{R}^2} u|^2 + |\nabla_{S^2} u|^2 \right] d\mu$$

is a norm on Y^1 . We denote W^1 the completion of Y^1 for the norm $\|\cdot\|_{W^1}$. The rotationally invariant fields in the $y - z$ plane play a peculiar role and we consider the subspaces

$$(IV.10) \quad X_0^0 := \left\{ u \in X^0, \quad y\partial_z u - z\partial_y u = 0 \right\}, \quad X_\perp^0 := (X_0^0)^\perp, \quad W_*^1 := W^1 \cap X_*^0, \quad * = 0, \perp.$$

Lemma IV.1. *$C_0^\infty(\Sigma)$ is dense in Y^1 and in W^1 . The embedding of Y^1 in X^0 is compact. W^1 is a subspace of X^0 , W_\perp^1 is a closed subspace of Y^1 and for all $u \in W_\perp^1$, it holds that*

$$(IV.11) \quad \int_{\Sigma} (r^2 + 1) |u|^2 d\mu \leq \int_{\Sigma} \frac{(r^2 + 1)^2}{(1 + \sqrt{r^2 + 1})^2} e^{2\sqrt{r^2 + 1}} |\nabla_{\mathbb{R}^2} u|^2 d\mu.$$

For all $u \in W_0^1$ we have:

$$(IV.12) \quad \int_{\Sigma} [1 + V(r)] |u|^2 d\mu \leq \int_{\Sigma} \frac{(r^2 + 1)^2}{(1 + \sqrt{r^2 + 1})^2} e^{2\sqrt{r^2 + 1}} |\nabla_{\mathbb{R}^2} u|^2 d\mu$$

where V is the positive potential defined by:

$$(IV.13) \quad V(r) := \frac{1}{4x^2} - \frac{1}{\sinh^2(2x)}, \quad r = \sinh x.$$

Proof. Given u in Y^1 we take some function $\chi \in C_0^\infty(\mathbb{R}^2)$ such that $\chi(y, z) = 1$ if $y^2 + z^2 \leq 1$ and $\chi(y, z) = 0$ if $y^2 + z^2 \geq 2$. To prove that $u_n(y, z, \Omega_2) := \chi\left(\frac{y}{n}, \frac{z}{n}\right) u(y, z, \Omega_2)$ tends to u in Y^1 as $n \rightarrow \infty$, it is sufficient to prove that

$$I_n := \frac{1}{n^2} \int_{\Sigma} \left| \nabla \chi \left(\frac{y}{n}, \frac{z}{n} \right) \right|^2 |u|^2 (r^2 + 1)^{\frac{3}{2}} dy dz d\Omega_2 \rightarrow 0, \quad n \rightarrow \infty.$$

Since

$$(IV.14) \quad e^{2r} \sim y^2 + z^2, \quad y^2 + z^2 \rightarrow \infty,$$

we get

$$I_n \lesssim \int_{\{n^2 \leq y^2 + z^2 \leq 2n^2\}} \int_{S^2} (r^2 + 1) |u|^2 d\mu \rightarrow 0, \quad n \rightarrow \infty,$$

and we deduce that the subset of the compactly supported functions of Y^1 is dense. Now if u is a function of Y^1 supported in $\{y^2 + z^2 \leq R^2\} \times S^2$, then u belongs to the classical Sobolev space $H^1(\Sigma)$ and it is well known that there exists a sequence $\varphi_n \in C_0^\infty(\Sigma)$ supported in $\{y^2 + z^2 \leq R^2 + 1\} \times S^2$ converging to u in H^1 . Since the H^1 norm and the Y^1 norm are equivalent on the space of the functions supported in a given compact, we conclude that φ_n tends to u in Y^1 .

To establish the compactness of the inclusion, we consider a sequence u_n weakly converging to zero in Y^1 and we denote $K := \sup_n \|u_n\|_{Y^1}$. Given $\epsilon > 0$, $R_\epsilon > 1$ we deduce from (IV.14) that

$$\int_{R_\epsilon^2 \leq y^2 + z^2} \int_{S^2} |u_n|^2 d\mu \lesssim \left(\frac{1}{\ln R_\epsilon} \right)^2 \int_{\Sigma} (r^2 + 1) |u_n|^2 d\mu \leq \left(\frac{K}{\ln R_\epsilon} \right)^2$$

hence we can fix $R_\epsilon > 1$ such that

$$\sup_n \int_{R_\epsilon^2 \leq y^2 + z^2} \int_{S^2} |u_n|^2 d\mu \leq \frac{\epsilon}{2}.$$

We choose $\chi \in C_0^\infty(\Sigma)$ such that $\chi(y, z, \Omega_2) = 1$ if $y^2 + z^2 \leq R_\epsilon^2$. We have

$$\int_{y^2 + z^2 \leq R_\epsilon^2} \int_{S^2} |u_n|^2 d\mu \lesssim \int_{\Sigma} |\chi u_n|^2 dy dz d\Omega_2.$$

Now the sequence χu_n is compactly supported and tends weakly to zero in the usual Sobolev space $H^1(\Sigma)$. Then χu_n tends to zero in $L^2(\Sigma)$ and we can find N_ϵ such that for any $n \geq N_\epsilon$ we have

$$\int_{y^2 + z^2 \leq R_\epsilon^2} \int_{S^2} |u_n|^2 d\mu \leq \frac{\epsilon}{2}.$$

We conclude that u_n tends strongly to zero in X^0 .

To prove (IV.12), it is sufficient to consider $u \in C_0^\infty(\Sigma)$. We expand $u(y, z, \cdot)$ on the basis of the spherical harmonics $Y_{l,m}(\Omega_2)$, $l \in \mathbb{N}$, $-l \leq m \leq l$, of $L^2(S^2)$:

$$u(y, z, \Omega_2) = \sum_{l,m} u_{l,m}(y, z) Y_{l,m}(\Omega_2), \quad u_{l,m} \in C_0^\infty(\mathbb{R}^2),$$

and we have to investigate

$$\int_{\mathbb{R}^2} (r^2 + 1)^{\frac{3}{2}} |\nabla_{y,z} u_{l,m}(y, z)|^2 dy dz - \int_{\mathbb{R}^2} |u_{l,m}(y, z)|^2 d\nu.$$

Since $u_{l,m}$ is compactly supported, the square root of the first integral, is a norm equivalent to the $H^1(\mathbb{R}^2)$ norm. On the other hand, $C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ is dense in $H^1(\mathbb{R}^2)$. Therefore it is sufficient to find V such that for all $\Phi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$,

$$(IV.15) \quad \int_{\mathbb{R}^2} [1 + V(r)] |\Phi(y, z)|^2 d\nu \leq \int_{\mathbb{R}^2} (r^2 + 1)^{\frac{3}{2}} |\nabla_{y,z} \Phi(y, z)|^2 dy dz.$$

We introduce a new radial coordinate x defined by

$$(IV.16) \quad x := \operatorname{arcsinh} r \in [0, \infty)$$

and we write

$$\Phi(y, z) = \frac{\phi(x, \psi)}{\sqrt{\sinh(x) \cosh(x)}}, \quad y = \frac{\sinh(x) e^{\cosh(x)}}{1 + \cosh(x)} \cos \psi, \quad z = \frac{\sinh(x) e^{\cosh(x)}}{1 + \cosh(x)} \sin \psi.$$

Some elementary computations show that (IV.15) is equivalent to

$$\int_0^{2\pi} \int_0^\infty V(\sinh x) |\phi|^2 dx d\psi \leq \int_0^{2\pi} \int_0^\infty \left| \frac{\partial \phi}{\partial x} \right|^2 - \frac{1}{\sinh^2(2x)} |\phi|^2 dx d\psi$$

for all $\phi \in C_0^\infty([0, \infty[\times S^1)$, V being chosen as (IV.13). This last inequality is a consequence of the Hardy inequality and the proof of (IV.12) is complete.

To establish the embedding of W_\perp^1 in Y^1 , we expand $u \in W^1$ in Fourier series with respect to ψ :

$$u(y, z, \Omega_2) = \sum_{n \in \mathbb{Z}} u_n(s, \Omega_2) e^{in\psi}, \quad s := \sqrt{y^2 + z^2},$$

and if $u \in X_\perp^0$ we have $u_0 = 0$, therefore we get if $u \in W_\perp^1$:

$$\int_{S^1} \frac{(r^2 + 1)^2}{(1 + \sqrt{r^2 + 1})^2} e^{2\sqrt{r^2 + 1}} |\nabla_{\mathbb{R}^2} u|^2 d\psi > \int_{S^1} (r^2 + 1) \sum_{n \in \mathbb{Z}^*} n^2 |u_n|^2 d\psi \geq \int_{S^1} (r^2 + 1) |u|^2 d\psi.$$

We deduce that (IV.11) is true on W_\perp^1 and that the norms Y^1 and W^1 are equivalent on W_\perp^1 . Hence the embedding of W_\perp^1 in Y^1 is proved.

Q.E.D.

We emphasize the drastic difference as regards the asymptotic behaviour at the space-like infinity, between, on the one hand, the case of the massless fields if $M = 0$ and $u \in W_0^1$, and on the other hand, the case of the massive fields if $M > 0$ or $M = 0$ and $u \in W_\perp^1$. In the sequel, we put

$$(IV.17) \quad X^1 := W^1 \text{ if } M = 0, \quad X^1 := Y^1 \text{ if } M > 0, \quad X_0^1 := X^1 \cap X_0^0, \quad X_\perp^1 := X^1 \cap X_\perp^0.$$

The classical fields, *i.e.* the fields that do not depend on the fifth Kaluza-Klein dimension ψ , belong to X_0^0 . A peculiar attention has to be paid to this case, mainly if $M = 0$. The Kaluza-Klein particles are described by the waves in X_\perp^0 and in some sense, these particles are massive, even if $M = 0$.

Theorem IV.2. *Given $f \in X^1$, $g \in X^0$, the Cauchy problem (IV.1), (IV.2) has a unique solution $u \in C^0(\mathbb{R}_t; X^1) \cap C^1(\mathbb{R}_t; X^0)$. Moreover the energy (IX.5) is decreasing as $|t| \rightarrow \infty$ and there exists a continuous function $C(t, t')$ independent of (f, g) such that*

$$(IV.18) \quad \|u(t)\|_{X^1} + \|\partial_t u(t)\|_{X^0} \leq C(t, t_0) (\|f\|_{X^1} + \|g\|_{X^0}).$$

Proof. To solve the Cauchy problem we could use the technics of Kato [48] or Lions [42], but we prefer the direct route using the well-posedness of the Cauchy problem in C_0^∞ by Leray [41] (see also [20]). We choose sequences f_n, g_n in $C_0^\infty(\Sigma)$ such that $f_n \rightarrow f$ in X^1 and $g_n \rightarrow g$ in X^0 as $n \rightarrow \infty$. According to Leray, since \mathcal{M} is globally hyperbolic, there exists a unique $u_n \in C^\infty(\mathcal{M})$ satisfying $u_n(t_0) = f_n$, $\partial_t u_n(t_0) = g_n$, and for any t , $u(t, \cdot)$ belongs to $C_0^\infty(\Sigma)$. Then u_n satisfies the energy estimate (IV.4) and by the Grönwall lemma, we deduce that u_n and $\partial_t u_n$ are Cauchy sequences respectively in $C^0(\mathbb{R}_t; X^1)$ and $C^0(\mathbb{R}_t; X^0)$ and tend to a solution $(u, \partial_t u)$. Also (IV.18) follows from (IV.4). To prove the uniqueness, we assume that $f = g = 0$, and we consider the solution v with initial data $v(t_1) = 0$, $\partial_t v(t_1) = \varphi$ for an arbitrary function $\varphi \in C_0^\infty(\Sigma)$. We put

$$F(t) := \langle u(t), \partial_t v(t) \rangle_\Sigma - \langle \partial_t u(t), v(t) \rangle_\Sigma$$

where $\langle \cdot, \cdot \rangle_\Sigma$ is the bracket of distributions on Σ . We have $F'(t) = \tanh(t)F(t)$ hence $F(t) = F(t_0) \cosh(t - t_0)$. Since $F(t_0) = 0$ we deduce that $0 = F(t_1) = \langle u(t_1), \varphi \rangle_\Sigma$, therefore $u = 0$.

Q.E.D.

We remark that ∂_ψ is a Killing vector, and if $\partial_\psi f = \partial_\psi g = 0$, then $\partial_\psi u = 0$ at any time, that is to say the Cauchy problem is well posed in (X_0^1, X_0^0) and in (X_\perp^1, X_\perp^0) as well. The solutions in X_0^0 describe the ordinary scalar waves which do not depend on the fifth dimension (see *e.g.* [5] to a presentation of the Kaluza-Klein theories). For such a smooth solution $u \in C^\infty(\mathcal{M})$, we have $\partial_y u = \partial_z u = 0$ at $y = z = 0$, or in term of (t, r, Ω_2, ψ) coordinates, $\partial_r u(t, 0, \Omega_2, \psi) = 0$. For this situation, the bubble \mathcal{B} can be interpreted as a wall that is a perfectly reflecting and expanding sphere.

V. SPECTRAL REPRESENTATIONS

The Witten spacetime \mathcal{M} can be described by

$$ds_{Witten}^2 = \cosh^2 x \left[dt^2 - dx^2 - \cosh^2 t d\Omega_2^2 - \frac{\sinh^2 x}{\cosh^4 x} d\Omega_1^2 \right], \quad t \in \mathbb{R}, \quad x \geq 0, \quad \Omega_d \in S^d.$$

Therefore \mathcal{M} is foliated by the two-parameter family of submanifolds defined by $x = Cst.$, $\psi = Cst.$, of De Sitter spacetimes dS^3 . In this section, we establish several analytic expressions of the scalar field as a superposition of Klein-Gordon waves on the 2+1-dimensional De Sitter spacetime

$$(V.1) \quad dS^3 := \mathbb{R}_t \times S^2, \quad g_{\alpha\beta} dx^\alpha dx^\beta = dt^2 - \cosh^2 t d\Omega_2^2,$$

called Kaluza-Klein tower. We sketch our strategy. We write the Klein-Gordon equation on \mathcal{M} as

$$(V.2) \quad \frac{1}{\cosh^2 t} \partial_t \left(\cosh^2 t \partial_t u \right) - \frac{1}{\cosh^2 t} \Delta_{S^2} u + Lu = 0,$$

where L is a differential operator on $\mathbb{R}_{y,z}^2$. We perform the complete spectral analysis of L and if $\Phi(\lambda; y, z)$ is a (generalized) eigenfunction of L satisfying $L\Phi = \lambda\Phi$ we can write

$$u(t, y, z, \Omega_2) = \int_{\sigma(L)} v_\lambda(t, \Omega_2) \Phi(\lambda; y, z) d\mu(\lambda)$$

where $d\mu$ is a spectral measure on the spectrum $\sigma(L)$ of L , and v_λ is solution of the Klein-Kordon equation with mass $\sqrt{\lambda}$ on dS^3 .

$$\frac{1}{\cosh^2 t} \partial_t \left(\cosh^2 t \partial_t v_\lambda \right) - \frac{1}{\cosh^2 t} \Delta_{S^2} v_\lambda + \lambda v_\lambda = 0.$$

We expand v_λ on the basis of the spherical harmonics as $v_\lambda(t, \Omega_2) = \frac{1}{\cosh t} \sum_{l,m} w_{l,m}(t) Y_{l,m}(\Omega_2)$ where $w_{l,m}$ is solution of

$$w'' + (\lambda - 1)w + \frac{l(l+1)}{\cosh^2 t} w = 0.$$

This equation being explicitly solvable in terms of Ferrers functions (see *e.g.* [25]), we obtain an analytic expression of u .

We now detail this approach. The operator L is given by

$$(V.3) \quad Lu := -\frac{e^{2\sqrt{r^2+1}} \sqrt{r^2+1}}{(1 + \sqrt{r^2+1})^2} \left[\partial_y \left((r^2+1)^{\frac{3}{2}} \partial_y u \right) + \partial_z \left((r^2+1)^{\frac{3}{2}} \partial_z u \right) \right] + M^2(r^2+1)u.$$

The measure ν being defined by $\nu = r dr d\psi$ or by (IV.6) in (y, z) coordinates, we consider L as a densely defined operator \mathbf{L}_* on H_* which is one of the following spaces

$$(V.4) \quad H^0 := L^2(\mathbb{R}^2, d\nu), \quad H_0^0 := \left\{ u \in H^0, y \partial_z u - z \partial_y u = 0 \right\}, \quad H_\perp^0 := \left(H_0^0 \right)^\perp,$$

endowed with its natural domain

$$(V.5) \quad D(\mathbf{L}_*) = \{ u \in H_*, q(u) < \infty, Lu \in H_* \}, \quad H_* = H^0, H_0^0, H_\perp^0,$$

where q is the quadratic form

$$(V.6) \quad \begin{aligned} q(u) &:= \int_{\mathbb{R}^2} \left[\frac{(r^2+1)^2}{(1 + \sqrt{r^2+1})^2} e^{2\sqrt{r^2+1}} |\nabla_{\mathbb{R}^2} u|^2 + M^2(r^2+1) |u|^2 \right] d\nu \\ &= \int_{\mathbb{R}^2} \left[(r^2+1)^{\frac{3}{2}} |\nabla_{\mathbb{R}^2} u|^2 + M^2(r^2+1)^{\frac{1}{2}} (1 + \sqrt{r^2+1})^2 e^{-2\sqrt{r^2+1}} |u|^2 \right] dy dz. \end{aligned}$$

On these spaces, \mathbf{L}_* is a positive self-adjoint operator and the domain $Q(q)$ of q can be identified with the closed subspace $\{ u \in X^1, \nabla_{S^2} u = 0 \} \cap X_*$, $X_* = X^0, X_0^0, X_\perp^0$. We deduce from (IV.12) and (IV.11) that $1 < \mathbf{L}$ if $M > 0$, and $1 < \mathbf{L}_\perp$, $1 \leq \mathbf{L}_0$ if $M = 0$. Hence the spectrum of these operators is included in $[1, \infty[$ and 1 is not an eigenvalue of \mathbf{L} if $M > 0$, and \mathbf{L}_\perp if $M = 0$. Moreover the resolvent of \mathbf{L} is compact if $M > 0$, and the resolvent of \mathbf{L}_\perp is compact if $M = 0$. We denote $(\lambda_k)_{k \in \mathbb{N}}$, $\lambda_k > 1$, the sequence of eigenvalues of \mathbf{L} for $M > 0$ (resp. \mathbf{L}_\perp if $M = 0$), and w_k a hilbertian basis of H^0 (resp. H_\perp^0) satisfying

$$(V.7) \quad Lw_k = \lambda_k w_k.$$

If $u \in C^0(\mathbb{R}_t, X^1) \cap C^1(\mathbb{R}_t, X^0)$ (resp. $u \in C^0(\mathbb{R}_t, X^1) \cap C^1(\mathbb{R}_t, X^0_\perp)$) is solution of the Klein-Gordon equation, we define for almost all $\Omega_2 \in S^2$

$$(V.8) \quad u_k(t, \Omega_2) := \int_{\mathbb{R}^2} u(t, y, z, \Omega_2) w_k(y, z) d\nu,$$

hence we have $u_k \in C^0(\mathbb{R}_t, H^1(S^2)) \cap C^1(\mathbb{R}_t, L^2(S^2))$ where $H^1(S^2)$ is the usual Sobolev space on S^2 and

$$(V.9) \quad u(t, y, z, \Omega_2) = \sum_{k \in \mathbb{N}} u_k(t, \Omega_2) w_k(y, z)$$

where the series is converging in $C^0(\mathbb{R}_t, X^1) \cap C^1(\mathbb{R}_t, X^0)$. Moreover u_k is solution of

$$(V.10) \quad \frac{1}{\cosh^2 t} \partial_t \left(\cosh^2 t \partial_t v \right) - \frac{1}{\cosh^2 t} \Delta_{S^2} v + \lambda v = 0,$$

that is just the Klein-Gordon equation with mass $\sqrt{\lambda} = \sqrt{\lambda_k}$ on the De Sitter space-time dS^3 , and the energy of u is the sum of the energies of u_k :

$$(V.11) \quad E(u, t) = \sum_{n \in \mathbb{N}} E_{\lambda_k}(u_k, t), \quad E_{\lambda_k}(u_k, t) := \int_{S^2} \left| \partial_t u_k(t) \right|^2 + \frac{1}{\cosh^2 t} \left| \nabla_{S^2} u_k(t) \right|^2 + \lambda_k \left| u_k(t) \right|^2 d\Omega_2.$$

The Cauchy problem for (V.10) is easily solved by using an expansion on the basis of spherical harmonics on $(Y_{l,m}(\Omega_2))$, $l \in \mathbb{N}$, $m \in \mathbb{Z}$, $|m| \leq l$. For $v \in C^0(\mathbb{R}_t, H^1(S^2)) \cap C^1(\mathbb{R}_t, L^2(S^2))$ solution of (V.10), we write:

$$(V.12) \quad v(t, \Omega_2) = \sum_{l,m} v_{l,m}(t) Y_{l,m}(\Omega_2)$$

where the series is converging in $C^0(\mathbb{R}_t, H^1(S^2)) \cap C^1(\mathbb{R}_t, L^2(S^2))$ and $v_{l,m}$ is solution of the differential equation

$$(V.13) \quad v'' + 2 \tanh(t) v' + \frac{l(l+1)}{\cosh^2 t} v + \lambda v = 0.$$

Since all the properties of the dynamics in dS^3 follow from this equation, we shall investigate it with some details in the next section. We introduce a new function w defined by

$$(V.14) \quad v(t) := \frac{w(t)}{\cosh t}.$$

Then v is solution of (V.13) iff w is solution of the Schrödinger equation with a Pöschl-Teller potential

$$(V.15) \quad w'' + (\lambda - 1)w + \frac{l(l+1)}{\cosh^2 t} w = 0.$$

Lemma V.1. *For any $\lambda > 1$, $l \in \mathbb{N}$, the set of the solutions of (V.15) is given by*

$$(V.16) \quad w(t) = A^+ \mathbf{P}_l^{i\sqrt{\lambda-1}}(\tanh t) + A^- \mathbf{P}_l^{i\sqrt{\lambda-1}}(-\tanh t), \quad A^\pm \in \mathbb{C},$$

where \mathbf{P}_l^μ are the Ferrers functions. A^\pm are linked with the Cauchy data $w(0)$, $w'(0)$ by the following

$$(V.17) \quad A^\pm = \frac{2^{-i\sqrt{\lambda-1}}}{2\sqrt{\pi}} \left[\Gamma\left(\frac{l}{2} + 1 - \frac{i}{2}\sqrt{\lambda-1}\right) \Gamma\left(\frac{1}{2} - \frac{l}{2} - \frac{i}{2}\sqrt{\lambda-1}\right) w(0) \right. \\ \left. \pm \frac{1}{2} \Gamma\left(\frac{l+1}{2} - \frac{i}{2}\sqrt{\lambda-1}\right) \Gamma\left(-\frac{l}{2} - \frac{i}{2}\sqrt{\lambda-1}\right) w'(0) \right].$$

Proof. We put $F(\tanh t) = w(t)$ and we check that w is solution of (V.15) iff $F(\xi)$ is solution of the associated Legendre equation

$$(V.18) \quad (1 - \xi^2)F''(\xi) - 2\xi F'(\xi) + \left(l(l+1) - \frac{1-\lambda}{1-\xi^2}\right)F(\xi) = 0, \quad -1 < \xi < 1.$$

We deduce that w is given by (V.16) where we refer to [43] for the notations and properties of the special functions, in particular, (V.17) follows from the formulas (14.5.1) and (14.5.2) of [43].

Q.E.D.

Finally we have obtained the analytic expression of the waves if the effective mass is not zero:

Theorem V.2. *Let u be a solution of (IV.1) in $C^0(\mathbb{R}_t, X^1) \cap C^1(\mathbb{R}_t, X^0)$. If $M = 0$, we assume that $u \in C^0(\mathbb{R}_t, X^1) \cap C^1(\mathbb{R}_t, X_\perp^0)$. Then there exists two sequences of complex numbers $A_{k,l,m}^\pm$ such that*

$$(V.19) \quad u(t, y, z, \Omega_2) = \sum_{k,l,m,\pm} \frac{1}{\cosh t} A_{k,l,m}^\pm P_l^{i\sqrt{\lambda_k-1}}(\pm \tanh t) w_k(y, z) Y_{l,m}(\Omega_2),$$

where $\lambda_k > 1$ is the sequence of eigenvalues of \mathbf{L} (resp. \mathbf{L}_\perp if $M = 0$) and the eigenfunctions w_k satisfy (V.7). Here $k, l \in \mathbb{N}$, $m \in \mathbb{Z}$, $-l \leq m \leq l$ and the series is converging in $C^0(\mathbb{R}_t, X^1) \cap C^1(\mathbb{R}_t, X^0)$.

To get the expression of u in term of a Kaluza-Klein tower, it will be convenient to use the spherical coordinates (t, r, Ω_2, ψ) introduced in Part two, for which the Klein-Gordon equation has the form:

$$(V.20) \quad \frac{1}{\cosh^2 t} \partial_t (\cosh^2 t \partial_t u) - \frac{1}{r} \partial_r (r(r^2 + 1) \partial_r u) - \frac{1}{\cosh^2 t} \Delta_{S^2} u - \frac{(r^2 + 1)^2}{r^2} \partial_\psi^2 u + M^2(r^2 + 1)u = 0.$$

Hence, replacing the (y, z) coordinates by the polar coordinates $(r, \psi) \in [0, \infty[\times [0, 2\pi[$, we have $d\nu = r dr d\psi$ and

$$(V.21) \quad L = -\frac{1}{r} \partial_r (r(r^2 + 1) \partial_r \cdot) - \frac{(r^2 + 1)^2}{r^2} \partial_\psi^2 + M^2(r^2 + 1).$$

For $n \in \mathbb{Z}$ we introduce the Hilbert subspaces

$$(V.22) \quad H_n^0 := \left\{ u \in H^0, \partial_\psi u = inu \right\}.$$

Then we have

$$H^0 = \bigoplus_{n \in \mathbb{Z}} H_n^0, \quad H_\perp^0 = \bigoplus_{n \in \mathbb{Z} \setminus \{0\}} H_n^0.$$

Operator L is reduced by these spaces and we have

$$L = \bigoplus_{n \in \mathbb{Z}} L_n$$

where

$$(V.23) \quad L_n := -\frac{1}{r} \frac{\partial}{\partial r} \left(r(r^2 + 1) \frac{\partial}{\partial r} \cdot \right) + n^2 \frac{(r^2 + 1)^2}{r^2} + M^2(r^2 + 1).$$

Therefore we know (see e.g. [52], Theorem 7.28) that L_n endowed with the domain $D(\mathbf{L}) \cap H_n^0$ is a selfadjoint operator \mathbf{L}_n on H_n^0 (see also an interpretation of these domains at the end of this part). Moreover if $(n, M) \neq (0, 0)$, $1 + n^2 + M^2 < \mathbf{L}_n$ and its resolvent is compact. Hence $\sigma(\mathbf{L}_n) = \sigma_p(\mathbf{L}_n) \subset]1 + n^2 + M^2, \infty[$ in this case, and we denote $(\lambda_{n,k})_{k \in \mathbb{N}}$ the sequence of its eigenvalues and $(w_{n,k}(r) e^{in\psi})_{k \in \mathbb{N}} \subset D(\mathbf{L}_n)$ a Hilbertian basis of H_n^0 satisfying

$$(V.24) \quad L_n (w_{n,k} \otimes e^{in\psi}) = \lambda_{n,k} w_{n,k} \otimes e^{in\psi}.$$

Then for all k there exists a unique (n, k') such that we obtain λ_k and $w_k(y, z)$ as $\lambda_k = \lambda_{n, k'}$, $w_k(y, z) = w_{n, k'}(r)e^{in\psi}$ where r, ψ and y, z are linked by (II.8), and conversely, for any (n, k') there exists a unique k such that λ_k and $w_k(y, z)$ given by the previous equalities are spectral quantities for \mathbf{L} . We may express the series (VI.21) converging in $C^0(\mathbb{R}_t, X^1) \cap C^1(\mathbb{R}_t, X^0)$ as

$$(V.25) \quad u(t, r, \Omega_2, \psi) = \sum_{k, n} \left[\sum_{l, m, \pm} \frac{1}{\cosh t} A_{k, l, m, n}^{\pm} P_l^{i\sqrt{\lambda_{n, k}-1}}(\pm \tanh t) Y_{l, m}(\Omega_2) \right] w_{n, k}(r) e^{in\psi},$$

where $w_{n, k}$ satisfies (V.24). Here $k, l \in \mathbb{N}$, $m \in \mathbb{Z}$, $-l \leq m \leq l$ and $n \in \mathbb{Z}$ if $M \neq 0$, $n \in \mathbb{Z} \setminus \{0\}$ if $M = 0$. We conclude that the Kaluza-Klein tower is given by the

Corollary V.3. *Let u be as in Theorem V.2. Then u can be represented as*

$$(V.26) \quad u(t, r, \Omega_2, \psi) = \sum_{n, k} U_{n, k}(t, \Omega_2) w_{n, k}(r) e^{in\psi},$$

where $w_{n, k}$ is the eigenfunction defined by (V.24) and

$$(V.27) \quad U_{n, k}(t, \Omega_2) = \sum_{l, m, \pm} \frac{1}{\cosh t} A_{k, l, m, n}^{\pm} P_l^{i\sqrt{\lambda_{n, k}-1}}(\pm \tanh t) Y_{l, m}(\Omega_2),$$

is solution of the Klein-Gordon equation (V.10) on dS^3 with the effective mass $\sqrt{\lambda} = \sqrt{\lambda_{n, k}}$.

To achieve the study of the massless case $M = 0$, $n = 0$, we have to consider the initial data $f \in W_0^1$, $g \in X_0^0$. The situation differs drastically from the previous one since the embedding of W_0^1 in X^0 is not compact and the spectrum of \mathbf{L}_0 is not discrete. For $u \in H_0^0$ we write $u(r)$ instead of $u(y, z)$ and we shall identify \mathbf{L}_0 with the operator

$$(V.28) \quad \mathfrak{L}_0 := -\frac{1}{r} \frac{d}{dr} \left(r(r^2 + 1) \frac{d}{dr} \right)$$

on the space

$$(V.29) \quad \mathfrak{h}_0 := L^2([0, \infty[_r, r dr)$$

endowed with the domain

$$(V.30) \quad D(\mathfrak{L}_0) := \{u \in \mathfrak{h}_0; \ u(y, z) \in D(\mathbf{L}_0)\}.$$

The key tool to represent the Klein-Gordon field, is the spectral resolution of this operator that is stated in the following proposition.

Proposition V.4. \mathfrak{L}_0 is a self-adjoint operator on \mathfrak{h}_0 and $1 \leq \mathfrak{L}_0$. Its domain is characterized by

$$(V.31) \quad D(\mathfrak{L}_0) = \left\{ u \in \mathfrak{h}_0; \ (r^2 + 1)^{\frac{1}{2}} u', \ \mathfrak{L}_0 u \in \mathfrak{h}_0, \ \lim_{r \rightarrow 0} u'(r) = 0 \right\}.$$

Its spectrum is absolutely continuous and equal to $[1, \infty[$. Its spectral resolution is given by the map $\mathcal{F} : u \mapsto \hat{u}$ which is an isometry from \mathfrak{h}_0 onto $L^2([1, \infty[_\lambda, d\lambda)$ defined by

$$(V.32) \quad \hat{u}(\lambda) = \lim_{A \rightarrow \infty} \int_0^A u(r) w(\lambda; r) r dr \text{ in } L^2([1, \infty[_\lambda, d\lambda),$$

where the generalized eigenfunction $w(\lambda; r)$ is given by

$$(V.33) \quad w(\lambda; r) = \frac{1}{\pi} \left(\frac{2}{r\sqrt{r^2 + 1}} \right)^{\frac{1}{2}} \tanh(\pi\sqrt{\lambda - 1}) Q_{-\frac{1}{2}}^{i\sqrt{\lambda - 1}}(\coth(2x))$$

where Q_ν^μ is the associated Legendre function of second kind. Moreover we have

$$(V.34) \quad u(r) = \lim_{A \rightarrow \infty} \int_1^A \hat{u}(\lambda) w(\lambda; r) d\lambda \text{ in } \mathfrak{h}_0$$

and

$$(V.35) \quad u \in D(\mathfrak{L}_0), \quad \mathcal{F}(\mathfrak{L}_0 u)(\lambda) = \lambda \hat{u}(\lambda).$$

Proof. Since \mathbf{L}_0 is self-adjoint and $1 \leq \mathbf{L}_0$, the same properties hold for \mathfrak{L}_0 that is unitarily equivalent with it. We use the radial coordinate x defined by (IV.16) for which

$$\mathfrak{L}_0 = -\frac{1}{\sinh(2x)} \frac{d}{dx} \left(\sinh(2x) \frac{d}{dx} \right).$$

Given $u \in H_0^0$, we write by abuse of notation $u(r) = u(x) = u(y, z)$, and if $u \in D(\mathbf{L}_0)$ we also write $\mathbf{L}_0 u(y, z) = \mathfrak{L}_0 u(x)$. Since $C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ is dense in $H^1(\mathbb{R}^2)$, we get that $C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ is dense in $Q(q)$ endowed with the norm $\sqrt{q(u) + \|u\|_{H^0}^2}$ where q is the form (V.6) with $M = 0$. We conclude that \mathbf{L}_0 is the Friedrichs extension of the operator L_0 endowed with the domain $C_0^\infty(\mathbb{R}^2 \setminus \{0\}) \cap H_0^0$. We introduce the isometry

$$(V.36) \quad T := u \mapsto v, \quad v(x) = \left(\frac{1}{2} \sinh(2x) \right)^{\frac{1}{2}} u(x),$$

from \mathfrak{h}_0 onto $L^2([0, \infty[, dx)$ with which the operator becomes

$$(V.37) \quad \mathbb{L}_0 := T \mathfrak{L}_0 T^{-1} = -\frac{d^2}{dx^2} - \frac{1}{\sinh^2(2x)} + 1.$$

Then \mathbb{L}_0 is the Friedrichs extension of the differential operator (V.37) endowed with the domain $C_0^\infty(0, \infty)$. We know (see e.g. [44], p. 104), that $v \in L^2(0, \infty)$ belongs to $D(\mathbb{L}_0)$ iff $\mathfrak{L}_0 v \in L^2$ and there exists $\varphi_n \subset C_0^\infty(0, \infty)$, such that $\varphi_n \rightarrow v$ in L^2 and

$$\lim_{n, m \rightarrow \infty} \int_0^\infty |\varphi'_n - \varphi'_m|^2 - \frac{1}{\sinh^2(2x)} |\varphi_n - \varphi_m|^2 dx = 0.$$

These constraints are equivalent with $v'' + (4x^2)^{-1}v \in L^2$ and

$$\lim_{n, m \rightarrow \infty} \int_0^\infty |\varphi'_n - \varphi'_m|^2 - \frac{1}{4x^2} |\varphi_n - \varphi_m|^2 dx = 0.$$

We conclude that $D(\mathbb{L}_0)$ is exactly the domain of the Friedrichs extension of the Bessel operator

$$\mathbb{M}_0 := -\frac{d^2}{dx^2} - \frac{1}{4x^2}$$

that is (see e.g. [29]):

$$(V.38) \quad D(\mathbb{M}_0) = \left\{ v \in L^2(0, \infty); v'' + (4x^2)^{-1}v \in L^2, \quad \lim_{x \rightarrow 0^+} \frac{1}{2}v(x)x^{-\frac{1}{2}} - v'(x)x^{\frac{1}{2}} = 0 \right\}.$$

We can obtain a more sharp characterization of this domain. We remark that for $0 < \epsilon < x$ we have

$$x^{\frac{1}{2}}v'(x) - \frac{1}{2}x^{-\frac{1}{2}}v(x) - \left(\epsilon^{\frac{1}{2}}v'(\epsilon) - \frac{1}{2}\epsilon^{-\frac{1}{2}}v(\epsilon) \right) = \int_\epsilon^x \xi^{\frac{1}{2}} \left[v''(\xi) + \frac{1}{4\xi^2}v(\xi) \right] d\xi,$$

hence taking the limit as $\epsilon \rightarrow 0$ and applying the Cauchy-Schwarz inequality we get

$$\left| x^{\frac{1}{2}}v'(x) - \frac{1}{2}x^{-\frac{1}{2}}v(x) \right| \leq \frac{x}{\sqrt{2}} \left(\int_0^x \left| v''(\xi) + \frac{1}{4\xi^2}v(\xi) \right|^2 d\xi \right)^{\frac{1}{2}} = o(x).$$

We deduce that $u(x) = v(x)/\sqrt{\sinh(2x)}$ satisfies $\lim_{x \rightarrow 0^+} u'(x) = 0$ and finally (V.31) is proved.

Now we have

$$(\mathbb{M}_0 + 1)^{-1} - \mathbb{L}_0^{-1} = (\mathbb{M}_0 + 1)^{-1} V \mathbb{L}_0^{-1}, \quad V(x) := \frac{1}{\sinh^2(2x)} - \frac{1}{4x^2}.$$

We prove that this operator is compact on $L^2(0, \infty)$. We consider a sequence $v_n \in L^2(0, \infty)$ that weakly tends to zero. Given $\epsilon > 0$ we choose $R_\epsilon > 0$ such that $\|V\|_{L^\infty(R_\epsilon, \infty)} \sup_n \|v_n\|_{L^2} < \epsilon/2$. On the other hand we know by Theorem 7.1 of [29] that $\frac{d}{dx}(\mathbb{M}_0 + 1)^{-1}v_n \in L^1(0, R_\epsilon)$ and by the closed graph theorem the map $v \mapsto \frac{d}{dx}(\mathbb{M}_0 + 1)^{-1}v$ is bounded from $L^2(0, \infty)$ to $L^1(0, R_\epsilon)$. We deduce that $\frac{d}{dx}(\mathbb{M}_0 + 1)^{-1}v_n$ tends weakly to zero in $L^1(0, R_\epsilon)$ and $\sup_n \|\frac{d}{dx}(\mathbb{M}_0 + 1)^{-1}v_n\|_{L^1(0, R_\epsilon)} < \infty$, hence $(\mathbb{M}_0 + 1)^{-1}v_n(x)$ tends to zero for any $x \in (0, R_\epsilon)$ and $\sup_n \|(\mathbb{M}_0 + 1)^{-1}v_n\|_{L^\infty(0, R_\epsilon)} < \infty$. Finally the dominated convergence theorem assures that $\|(\mathbb{M}_0 + 1)^{-1}v_n\|_{L^2(0, R_\epsilon)} < \frac{\epsilon}{2}$ for n large enough and the proof is complete. Then we conclude by the Weyl theorem that \mathbb{L}_0 and $\mathbb{M}_0 + 1$ have the same essential spectrum that is $[1, \infty[$ ([29]).

To pursue the spectral analysis of \mathbb{L}_0 we investigate the equation

$$(V.39) \quad -\frac{d^2v}{dx^2} - \frac{1}{\sinh^2(2x)}v + v = \lambda v,$$

satisfying the boundary condition

$$(V.40) \quad \lim_{x \rightarrow 0^+} \frac{1}{2}v(x)x^{-\frac{1}{2}} - v'(x)x^{\frac{1}{2}} = 0.$$

We introduce a new coordinate

$$(V.41) \quad X := \frac{2r^2 + 1}{2r\sqrt{r^2 + 1}} = \coth(2x) \in]1, \infty[.$$

Then (V.39) is equivalent to

$$(V.42) \quad (1 - X^2)\frac{d^2v}{dX^2} - 2X\frac{dv}{dX} - \left(\frac{1}{4} + \frac{\frac{1}{4} - \frac{\lambda}{4}}{1 - X^2}\right)v = 0.$$

We recognize the associated Legendre equation with $\nu = -\frac{1}{2}$ and $\mu^2 = \frac{1-\lambda}{4}$ hence we get that the solutions of (V.39) are

$$(V.43) \quad v(x) = AP_{-\frac{1}{2}}^{-\mu}(\coth(2x)) + BQ_{-\frac{1}{2}}^{\mu}(\coth(2x)), \quad \mu^2 = \frac{1-\lambda}{4}, \quad \Re\mu \geq 0, \quad A, B \in \mathbb{C},$$

where $P_{\nu}^{-\mu}$ and Q_{ν}^{μ} are the usual associated Legendre functions of first and second kind (notations of [43]). To take into account the condition at zero, we use the following asymptotics at the infinity of the Legendre functions and their derivatives (see [43], using formulas 14.8.15 and 14.10.4, 14.3.10)

$$(V.44) \quad Q_{-\frac{1}{2}}^{\mu}(X) \sim \sqrt{\frac{\pi}{2X}}, \quad \frac{d}{dX}Q_{-\frac{1}{2}}^{\mu}(X) \sim -\frac{1}{2X}\sqrt{\frac{\pi}{2X}}, \quad X \rightarrow \infty, \quad \mu \in \mathbb{C}.$$

Given any complex number μ and some $\chi \in C_0^\infty(\mathbb{R})$, $\chi(x) = 1$ for $|x| \leq 1$, we put

$$(V.45) \quad \mathbf{v}_\mu(x) := \chi(x)Q_{-\frac{1}{2}}^{\mu}(\coth(2x)), \quad x \in (0, \infty).$$

Therefore \mathbf{v}_μ belongs to the domain $D(\mathbb{M}_0) = D(\mathbb{L}_0)$ and the boundary condition (V.40) defining this domain is equivalent to

$$(V.46) \quad \lim_{x \rightarrow 0^+} v(x)\mathbf{v}'_\mu(x) - v'(x)\mathbf{v}_\mu(x) = 0$$

for some, hence for any, $\mu \in \mathbb{C}$. Since the wronskian of $P_{-\frac{1}{2}}^{-\mu}(X)$ and $Q_{-\frac{1}{2}}^{\mu}(X)$ is $1/(\Gamma(\mu + \frac{1}{2})(1 - X^2))$ we conclude that the solutions of (V.39) satisfying (V.40) are:

$$(V.47) \quad v(x) = A Q_{-\frac{1}{2}}^{\mu}(\coth(2x)), \quad A \in \mathbb{C}, \quad \mu^2 = \frac{1 - \lambda}{4}.$$

To investigate the behaviour at the infinity we recall that

$$(V.48) \quad Q_{-\frac{1}{2}}^0(X) \sim -\frac{1}{2\sqrt{\pi}} \ln(X - 1), \quad X \rightarrow 1^+,$$

$$(V.49) \quad P_{\nu}^{-\mu}(X) \sim \frac{1}{\Gamma(\mu + 1)} \left(\frac{X - 1}{2} \right)^{\frac{\mu}{2}}, \quad X \rightarrow 1^+, \quad -\mu \notin \mathbb{N}^*,$$

and

$$(V.50) \quad Q_{-\frac{1}{2}}^{\mu}(X) = \frac{\pi}{2 \sin(\mu\pi)} \left[\frac{P_{-\frac{1}{2}}^{\mu}(X)}{\Gamma\left(\frac{1}{2} + \mu\right)} - \frac{P_{-\frac{1}{2}}^{-\mu}(X)}{\Gamma\left(\frac{1}{2} - \mu\right)} \right], \quad \mu \notin \mathbb{Z}.$$

We get that for $\lambda > 1$ we have

$$(V.51) \quad Q_{-\frac{1}{2}}^{\frac{i}{2}\sqrt{\lambda-1}}(X) \sim \frac{\pi}{2i \sinh\left(\frac{\pi}{2}\sqrt{\lambda-1}\right)} \left[\frac{1}{\Gamma\left(\frac{1}{2} + \frac{i}{2}\sqrt{\lambda-1}\right) \Gamma\left(1 - \frac{i}{2}\sqrt{\lambda-1}\right)} \left(\frac{X-1}{2}\right)^{-\frac{i}{4}\sqrt{\lambda-1}} \right. \\ \left. - \frac{1}{\Gamma\left(\frac{1}{2} - \frac{i}{2}\sqrt{\lambda-1}\right) \Gamma\left(1 + \frac{i}{2}\sqrt{\lambda-1}\right)} \left(\frac{X-1}{2}\right)^{\frac{i}{4}\sqrt{\lambda-1}} \right], \quad X \rightarrow 1^+.$$

$$(V.52) \quad Q_{-\frac{1}{2}}^{\frac{i}{2}\sqrt{\lambda-1}}(\coth(2x)) \sim \frac{\pi}{2i \sinh\left(\frac{\pi}{2}\sqrt{\lambda-1}\right)} \left[\frac{e^{ix\sqrt{\lambda-1}}}{\Gamma\left(\frac{1}{2} + \frac{i}{2}\sqrt{\lambda-1}\right) \Gamma\left(1 - \frac{i}{2}\sqrt{\lambda-1}\right)} \right. \\ \left. - \frac{e^{-ix\sqrt{\lambda-1}}}{\Gamma\left(\frac{1}{2} - \frac{i}{2}\sqrt{\lambda-1}\right) \Gamma\left(1 + \frac{i}{2}\sqrt{\lambda-1}\right)} \right], \quad x \rightarrow +\infty,$$

and

$$(V.53) \quad Q_{-\frac{1}{2}}^0(\coth(2x)) \sim -\frac{2}{\sqrt{\pi}} x, \quad x \rightarrow +\infty.$$

We conclude that the point spectrum of \mathbb{L}_0 is empty.

To prove that the spectrum is absolutely continuous and to construct the spectral representation, we employ the technics of Pearson [44]. First, $x = 0$ is in the limit-circle case and $x = \infty$ is in the limit-point case. From (V.49) and (V.48), we get that any solution $v \neq 0$ of (V.39) has the asymptotics as $x \rightarrow \infty$: $|v(x)| \sim a_+ e^{2ix\sqrt{\lambda-1}} + a_- e^{-2ix\sqrt{\lambda-1}}$, $a_{\pm} \in \mathbb{C}$, $(a_+, a_-) \neq (0, 0)$, hence $\|v\|_{L^2(0, R)} \sim c R^{\frac{1}{2}}$, $c > 0$. We conclude that there is no solution of (V.39) that is sequentially subordinate at $x = \infty$. Now given $\lambda_0 \geq 1$, and $\lambda = \lambda_0 + i\epsilon$, $0 < \epsilon$, estimate (V.49) shows that any solution of (V.39) square integrable for large x has the form $A P_{\nu}^{-\mu}(\coth(2x))$ with $4\mu^2 = \lambda_0 - 1 + i\epsilon$, $\Re \mu > 0$. We deduce that $f_+(x) := A P_{-\frac{1}{2}}^{-\frac{i}{2}\sqrt{\lambda-1}}(\coth(2x))$, $A \in \mathbb{C}$, is an upper solution of (V.39). To normalize this function we evaluate the Wronskian

$$P_{\nu}^{-\mu}(X) \overline{\frac{dP_{\nu}^{-\mu}}{dX}(X)} - \overline{P_{\nu}^{-\mu}(X)} \frac{dP_{\nu}^{-\mu}}{dX}(X) \sim \frac{1}{1 - X^2} \left| \left(\frac{X - 1}{2} \right)^{\mu} \right| \frac{2i \Im \mu}{|\Gamma(\mu + 1)|^2}, \quad X \rightarrow 1^+, \quad \mu \notin \mathbb{N}^*.$$

We deduce that for $\lambda > 1$

$$f_+ \overline{f'_+} - \overline{f_+} f'_+ = -\frac{4A \sinh(\frac{\pi}{2}\sqrt{\lambda-1})}{\pi} i.$$

We look for A such that $f_+ \overline{f'_+} - \overline{f_+} f'_+ = -i$, hence we obtain the normalized upper solution

$$(V.54) \quad f_+(x) := \frac{\pi}{4 \sinh(\frac{\pi}{2}\sqrt{\lambda-1})} P_{-\frac{1}{2}}^{-\frac{i}{2}\sqrt{\lambda-1}}(\coth(2x)).$$

Finally the spectral function is defined as the solution $v(\lambda; x)$ of (V.39) and (V.40) such as its spectral amplitude $\mathcal{A} := 2 | f_+ \partial_x v - v \partial_x f_+ |$ is equal to 1. Using (V.47) and the Wronskian relation (14.2.8) of [43] we obtain

$$(V.55) \quad v(\lambda; x) = \frac{\tanh\left(\frac{\pi}{2}\sqrt{\lambda-1}\right)}{\sqrt{\pi}} Q_{-\frac{1}{2}}^{\frac{i}{2}\sqrt{\lambda-1}}(\coth(2x)).$$

The Theorem 7.4 of [44]) assures that the spectrum of \mathbb{L}_0 is absolutely continuous and the map

$$v \in L^2(0, \infty) \mapsto \tilde{v}(\lambda) := \frac{\sqrt{2}}{\pi} \lim_{R \rightarrow \infty} \int_0^R v(x) \tanh\left(\frac{\pi}{2}\sqrt{\lambda-1}\right) Q_{-\frac{1}{2}}^{\frac{i}{2}\sqrt{\lambda-1}}(\coth(2x)) dx \text{ in } L^2(1, \infty),$$

is an isometry from $L^2(0, \infty)$ onto $L^2(1, \infty)$ satisfying

$$v(x) = \frac{\sqrt{2}}{\pi} \lim_{R \rightarrow \infty} \int_1^R \tilde{v}(\lambda) \tanh\left(\frac{\pi}{2}\sqrt{\lambda-1}\right) Q_{-\frac{1}{2}}^{\frac{i}{2}\sqrt{\lambda-1}}(\coth(2x)) d\lambda \text{ in } L^2(0, \infty),$$

$$v \in D(\mathbb{L}_0), \quad \widetilde{\mathbb{L}_0 v}(\lambda) = \lambda \tilde{v}(\lambda).$$

Returning to $u(r) = \left(\frac{1}{2} \sinh(2x)\right)^{-\frac{1}{2}} v(x)$ we obtain the spectral representation of \mathfrak{L}_0 .

Q.E.D.

We are ready to obtain the analytic form of the massless waves:

Theorem V.5. *We assume $M = 0$. Let $u \in C^0(\mathbb{R}_t; W_0^1) \cap C^1(\mathbb{R}_t; X_0^0)$ be a solution of (IV.1). Then there exists $\hat{u} \in C^0(\mathbb{R}_t; L^2([1, \infty[_\lambda; H^1(S^2))) \cap C^1(\mathbb{R}_t; L^2([1, \infty[_\lambda \times S^2)))$ such that*

$$(V.56) \quad u(t, r, \Omega_2) = \frac{1}{\pi} \lim_{A \rightarrow \infty} \left(\frac{2}{r\sqrt{r^2+1}} \right)^{\frac{1}{2}} \int_1^A \hat{u}(\lambda; t, \Omega_2) \tanh\left(\frac{\pi}{2}\sqrt{\lambda-1}\right) Q_{-\frac{1}{2}}^{\frac{i}{2}\sqrt{\lambda-1}}\left(\frac{2r^2+1}{2r\sqrt{r^2+1}}\right) d\lambda$$

where the limit holds in $C^0(\mathbb{R}_t; W_0^1) \cap C^1(\mathbb{R}_t; X_0^0)$. Moreover for almost all $\lambda > 1$, $\hat{u}(\lambda; \cdot)$ is solution of the Klein-Gordon equation with mass $\sqrt{\lambda}$ on the De Sitter space dS^3 . \hat{u} is given by the formula

$$(V.57) \quad \hat{u}(\lambda; t, \Omega_2) = \frac{1}{\cosh t} \sum_{l, m, \pm} A_{l, m}^\pm(\lambda) P_l^{i\sqrt{\lambda-1}}(\pm \tanh t) Y_{l, m}(\Omega_2),$$

where the limit holds in $C^0(\mathbb{R}_t; H^1(S^2)) \cap C^1(\mathbb{R}_t; L^2(S^2))$ with

$$(V.58) \quad A_{l, m}^\pm(\lambda) = \frac{2^{-i\sqrt{\lambda-1}}}{2\sqrt{\pi}} \left[\Gamma\left(\frac{l}{2} + 1 - \frac{i}{2}\sqrt{\lambda-1}\right) \Gamma\left(\frac{1}{2} - \frac{l}{2} - \frac{i}{2}\sqrt{\lambda-1}\right) w_{l, m}^0(\lambda) \right. \\ \left. \pm \frac{1}{2} \Gamma\left(\frac{l+1}{2} - \frac{i}{2}\sqrt{\lambda-1}\right) \Gamma\left(-\frac{l}{2} - \frac{i}{2}\sqrt{\lambda-1}\right) \right] w_{l, m}^1(\lambda),$$

(V.59)

$$w_{l, m}^k(\lambda) := \frac{1}{\pi} \lim_{A \rightarrow \infty} \tanh\left(\frac{\pi}{2}\sqrt{\lambda-1}\right) \int_0^A \int_{S^2} \left(\frac{2}{r\sqrt{r^2+1}} \right)^{\frac{1}{2}} \partial_t^k u(0, r, \Omega_2) Q_{-\frac{1}{2}}^{\frac{i}{2}\sqrt{\lambda-1}}\left(\frac{2r^2+1}{2r\sqrt{r^2+1}}\right) \overline{Y_{l, m}(\Omega_2)} r dr d\Omega_2,$$

where the limit holds in $L^2([1, \infty[_\lambda)$.

Proof. We note that for $u \in W_0^1$ we have:

$$\|u\|_{W_0^1}^2 = 2\pi \int_0^\infty \int_{S^2} \left[(r^2 + 1) |\partial_r u|^2 + |\nabla_{S^2} u|^2 \right] r dr d\Omega_2 = 2\pi \int_{S^2} \|\mathfrak{L}_0^{\frac{1}{2}} u(\cdot, \Omega_2)\|_{\mathfrak{h}_0}^2 + \|\nabla_{S^2} u(\cdot, \Omega_2)\|_{\mathfrak{h}_0}^2 d\Omega_2.$$

Given $u \in C^1(\mathbb{R}_t, X_0^0) \cap C^0(\mathbb{R}_t; W_0^1)$, for any $t \in \mathbb{R}$, the map $r \mapsto u(t, r, \Omega_2)$ belongs to \mathfrak{h}_0 for almost all $\Omega_2 \in S^2$. We denote

$$\hat{u}(t, \lambda, \Omega_2) := \mathcal{F}(u(t, \cdot, \Omega_2))(\lambda) = \lim_{A \rightarrow \infty} \int_0^A u(t, r, \Omega_2) w(\lambda; r) r dr, \text{ in } L^2([1, \infty[_\lambda, d\lambda)$$

where w is the spectral function (V.33). Since \mathcal{F} is an isometry, the map $u \mapsto (\hat{u}, \partial_t \hat{u}, \sqrt{\lambda} \hat{u}, \nabla_{S^2} \hat{u})$ is continuous from $C^1(\mathbb{R}_t, X_0^0) \cap C^0(\mathbb{R}_t; W_0^1)$ to $C^0(\mathbb{R}_t; L^2([1, \infty[_\lambda \times S_{\Omega_2}^2, d\lambda d\Omega_2))$ and

$$u(t, r, \Omega_2) = \lim_{A \rightarrow \infty} \int_1^A \hat{u}(t, \lambda, \Omega_2) w(\lambda; r) d\lambda$$

where the limit holds in $C^1(\mathbb{R}_t, X_0^0) \cap C^0(\mathbb{R}_t; W_0^1)$. Now we prove that if u is solution of the Klein-Gordon equation (IV.1) on the Witten space-time, then \hat{u} is solution of the Klein-Gordon equation with mass $\sqrt{\lambda}$ on the (2+1)-dimensional De Sitter space-time (V.10) in $\mathcal{D}'(\mathbb{R}_t \times S^2 \times]1, \infty[_\lambda)$. Due to the previous continuity for $u \mapsto \hat{u}$, it is sufficient to consider the case of the smooth solutions $u \in C^\infty(\mathbb{R}_t; C_0^\infty(\Sigma))$. Then $\hat{u} \in C^\infty(\mathbb{R}_t \times [1, \infty[_\lambda; L^2(S^2))$ and applying the transform \mathcal{F} to (IV.1) we can see that \hat{u} is solution of (V.10) on $\mathbb{R}_t \times S^2 \times [1, \infty[_\lambda$. Then we expand $\hat{u}(t, \lambda, \cdot)$ on the basis of spherical harmonics like in (V.12), and we conclude with Lemma V.1.

Q.E.D.

We end this part by a remark concerning the domain $D(\mathbf{L}) \cap H_n^0$ of the operators \mathbf{L}_n . The situation is similar to that of the usual Laplacian in polar coordinates, for which the zero mode satisfies a Neumann condition at the origin, the other modes a Dirichlet condition. If $n = 0$, the proof of the characterization (V.31) in the massless case holds again if $M > 0$, and we have $\lim_{r \rightarrow 0} u'(0) = 0$ for $u \in D(\mathbf{L}_0)$ whatever the mass $M \geq 0$. If $n \in \mathbb{Z}^*$, we use the isometry (V.36) to write $\mathbb{L}_n := T^{-1} \mathbf{L}_n T = -\frac{d^2}{dx^2} + V(x)$ with $V(x) = n^2 \frac{\cosh^4 x}{\sinh^2 x} - \frac{1}{\sinh^2(2x)} + 1 + M^2 \cosh^2 x$. If $n \neq 0$ we have $V(x) \geq \frac{3}{4x^2}$ and a classical result (see Theorem X.10 in [45]) assures that \mathbb{L}_n is essentially self-adjoint on $C_0^\infty([0, \infty[)$. We deduce that $u(0) = 0$ for $u \in D(\mathbf{L}_n)$, $n \neq 0$, $M \geq 0$.

VI. ASYMPTOTICS

In the previous section we have proved that the scalar fields u on the Witten universe can be expressed as a superposition of Klein-Gordon fields v_λ with mass $\sqrt{\lambda}$, on the De Sitter spacetime dS^3 :

$$u(t, x, \Omega_2, \psi) = \int_{\sigma(L)} v_\lambda(t, \Omega_2) \Phi(\lambda; x, \psi) d\mu(\lambda).$$

The spectrum $\sigma(L)$ is continuous in the massless case $M = 0$, $\partial_\psi u = 0$, and $\sigma(L)$ is discrete in the massive case. Therefore to investigate the asymptotic behaviours of u as $|t| \rightarrow \infty$, we have to study the behaviours of the solutions of the massive Klein-Kordon fields on dS^3 . Using the spherical harmonics expansion, we write v_λ as

$$v_\lambda(t, \Omega_2) = \frac{1}{\cosh t} \sum_{l,m} w_{l,m,\lambda}(t) Y_{l,m}(\Omega_2),$$

where $w_{l,m,\lambda}$ is solution of a Schrödinger equation with the Pöschl-Teller potential $\cosh^{-2} t$, that is explicitly solvable in terms of Ferrers functions. The key result on the asymptotic behaviour of the waves in dS^3 is given in Lemma VI.2 below that states

$$w_{l,m,\lambda}(t) \sim w_{in(out)}^+ e^{it\sqrt{\lambda}} + w_{in(out)}^- e^{-it\sqrt{\lambda}}, \quad t \rightarrow -(+)\infty,$$

and there is no mixing between the positive and negative frequencies:

$$w_{out}^{\pm} = (-1)^l \frac{\Gamma(1 \pm i\sqrt{\lambda-1})}{\Gamma(1 \mp i\sqrt{\lambda-1})} \frac{\Gamma(l+1 \mp i\sqrt{\lambda-1})}{\Gamma(l+1 \pm i\sqrt{\lambda-1})} w_{in}^{\pm}.$$

We deduce the fundamental result on the asymptotics of the fields in the Witten spacetime (Theorem VI.3): $(\cosh t)u(t)$ is asymptotically quasi-periodic in the massive case, and in contrast, it is dispersive in the massless case.

Due to the expansion of the bubble, all the fields are exponentially damped. Therefore it is natural to introduce the profile v of a field u defined by $v(t, \cdot) := \cosh(t)u(t, \cdot)$. Then u is solution of the Klein-Gordon equation (IV.1) iff v is solution of

$$(VI.1) \quad \left[\partial_t^2 - \frac{1}{\sinh(2x)} \partial_x (\sinh(2x) \partial_x) - \frac{1}{\cosh^2 t} \Delta_{S^2} - \frac{\cosh^4 x}{\sinh^2 x} \partial_\psi^2 + M^2 \cosh^2 x - 1 \right] v = 0,$$

where we use the x variable introduced in (IV.16), $x = \operatorname{arcsinh} r$. It is clear from Theorem IV.2 that the Cauchy problem for this equation is well posed in (X^1, X^0) , (X_\perp^1, X_\perp^0) , and (X_0^1, X_0^0) . Moreover there exists a natural energy, that is decreasing as $|t| \rightarrow \infty$

$$(VI.2) \quad E(v, t) := \int_{\Sigma_t} \left[|\partial_t v|^2 + |\partial_x v|^2 + \frac{1}{\cosh^2 t} |\nabla_{S^2} v|^2 + \frac{\cosh^4 x}{\sinh^2 x} |\partial_\psi v|^2 + (M^2 \cosh^2 x - 1) |v|^2 \right] d\mu$$

where the measure μ (IV.6) on Σ is now given by

$$d\mu = \frac{1}{2} \sinh(2x) \sin \theta \, dx d\theta d\varphi d\psi, \quad (x, \theta, \varphi, \psi) \in [0, \infty[\times [0, \pi] \times [0, 2\pi]^2.$$

The aim of this part consists in investigating the asymptotic behaviour of v as $t \rightarrow -(+)\infty$, by comparing $v(t)$ with the solutions $v_{in(out)}$ of

$$(VI.3) \quad \left[\partial_t^2 - \frac{1}{\sinh(2x)} \partial_x (\sinh(2x) \partial_x) - \frac{\cosh^4 x}{\sinh^2 x} \partial_\psi^2 + M^2 \cosh^2 x - 1 \right] v = 0,$$

for which the natural energy is

$$(VI.4) \quad E_\infty(v, t) := \int_{\Sigma_t} \left[|\partial_t v|^2 + |\partial_x v|^2 + \frac{\cosh^4 x}{\sinh^2 x} |\partial_\psi v|^2 + (M^2 \cosh^2 x - 1) |v|^2 \right] d\mu.$$

The inequalities (IV.11) and (IV.12) assure that, despite the term $-|v|^2$, the energies $E(v, t)$ and $E_\infty(v, t)$ are positive definite quadratic forms. We introduce the functional space associated to E_∞ ,

$$(VI.5) \quad \dot{X}^1 := L^2(S^2; Q(q))$$

where, given $M \geq 0$, $Q(q)$ is the form domain of the quadratic form q defined by (V.6), *i.e.* \dot{X}^1 is the closure of $C_0^\infty(\Sigma)$ for the norm

$$\|v\|_{\dot{X}^1}^2 = \int_{\Sigma} \left[|\partial_x v|^2 + \frac{\cosh^4 x}{\sinh^2 x} |\partial_\psi v|^2 + M^2 \cosh^2 x |v|^2 \right] d\mu.$$

We remark that the estimates (IV.11) and (IV.12) of Lemma IV.1 imply that

$$\|v\|_{X^0} \leq \|v\|_{\dot{X}^1}$$

even if $M = 0$.

Proposition VI.1. *Given $v_0 \in \dot{X}^1$, $v_1 \in X^0$, there exists a unique solution $v \in C^0(\mathbb{R}_t; \dot{X}^1) \cap C^1(\mathbb{R}_t; X^0)$ solution of (VI.3) with $v(0) = v_0$, $\partial_t v(0) = v_1$. The energy (VI.4) is positive and conserved. If $M > 0$ or if $M = 0$ and $v_j \in X_{\perp}^0$, we can write the field as*

$$(VI.6) \quad v(t, x, \Omega_2, \psi) = \sum_{k,l,m,n,\pm} A_{k,l,m,n}^{\pm} e^{\pm it\sqrt{\lambda_{n,k}-1}} w_{n,k}(\sinh x) Y_{l,m}(\Omega_2) e^{i\psi},$$

and if $M = 0$ and $v_j \in X_0^0$ we have

$$(VI.7) \quad v(t, x, \Omega_2) = \frac{2}{\pi} \lim_{A \rightarrow \infty} \left(\frac{1}{\sinh(2x)} \right)^{\frac{1}{2}} \int_1^A \sum_{l,m,\pm} A_{l,m}^{\pm}(\lambda) e^{\pm it\sqrt{\lambda-1}} Y_l^m(\Omega_2) \tanh\left(\frac{\pi}{2}\sqrt{\lambda-1}\right) Q_{-\frac{1}{2}}^{\frac{i}{2}\sqrt{\lambda-1}}(\coth(2x)) d\lambda$$

where the limits (VI.6) and (VI.7) hold in $C^0(\mathbb{R}_t; \dot{X}^1) \cap C^1(\mathbb{R}_t; X^0)$.

Proof. The existence and uniqueness of the Cauchy problem follow from classic results. For instance, we can apply Theorem 8.1, chap. 3, p. 287 of [42]. The assumptions (8.1) and (8.2) of this theorem are satisfied with $H = \dot{X}^1$, $V = X^0$, $a(v, v') = \langle v, v' \rangle_{\dot{X}^1} - \langle v, v' \rangle_{X^0}$, $\lambda = 1$, $\alpha = 1$. To establish the spectral expansions (VI.6) and (VI.7), we mimic the proofs of the Corollary V.3 and Theorem V.5, by replacing the differential equation (V.15) by the simpler harmonic oscillator $w'' + (\lambda - 1)w = 0$.

Q.E.D.

The root of the properties of the asymptotic behaviours of the waves in the Witten space-time, is given by the following lemma describing the dynamics of the scalar fields in the De Sitter space dS^3 .

Lemma VI.2. *Given $\lambda > 1$, $l \in \mathbb{N}$, $A^{\pm} \in \mathbb{C}$, we consider $w(t) := A^+ P_l^{i\sqrt{\lambda-1}}(\tanh t) + A^- P_l^{i\sqrt{\lambda-1}}(-\tanh t)$. Then there exist $w_{in(out)}^{\pm} \in \mathbb{C}$ such that:*

$$(VI.8) \quad \left| w(t) - \left(w_{in(out)}^+ e^{it\sqrt{\lambda-1}} + w_{in(out)}^- e^{-it\sqrt{\lambda-1}} \right) \right| \rightarrow 0, \quad t \rightarrow -\infty \quad (t \rightarrow +\infty),$$

$$(VI.9) \quad \left| w'(t) - i\sqrt{\lambda-1} \left(w_{in(out)}^+ e^{it\sqrt{\lambda-1}} - w_{in(out)}^- e^{-it\sqrt{\lambda-1}} \right) \right| \rightarrow 0, \quad t \rightarrow -\infty \quad (t \rightarrow +\infty),$$

where

$$(VI.10) \quad w_{in(out)}^{-(+)} = \frac{1}{\Gamma(1 - i\sqrt{\lambda-1})} A^{-(+)}, \quad w_{in(out)}^{+(-)} = (-1)^l \frac{\Gamma(l+1 + i\sqrt{\lambda-1})}{\Gamma(l+1 - i\sqrt{\lambda-1})\Gamma(1 + i\sqrt{\lambda-1})} A^{+(-)},$$

$$(VI.11) \quad w_{out}^{\pm} = (-1)^l \frac{\Gamma(1 \pm i\sqrt{\lambda-1})}{\Gamma(1 \mp i\sqrt{\lambda-1})} \frac{\Gamma(l+1 \mp i\sqrt{\lambda-1})}{\Gamma(l+1 \pm i\sqrt{\lambda-1})} w_{in}^{\pm}.$$

The energy of w defined by

$$E(w; t) := \frac{1}{2} |w'(t)|^2 + \frac{1}{2} \left(\lambda - 1 + \frac{l(l+1)}{\cosh^2 t} \right) |w(t)|^2$$

is a decreasing function as $|t| \rightarrow \infty$ and satisfies

(VI.12)

$$\begin{aligned} E(w, \infty) &:= \lim_{|t| \rightarrow \infty} E(w; t) = (\lambda - 1) \left[|w_{in/out}^+|^2 + |w_{in/out}^-|^2 \right] \\ &= \frac{\sqrt{\lambda - 1}}{\pi} \sinh(\pi\sqrt{\lambda - 1}) \left(|A^+|^2 + |A^-|^2 \right) \\ &= 2\sqrt{\lambda - 1} \sinh(\pi\sqrt{\lambda - 1}) \left[\left| \frac{\Gamma\left(\frac{l}{2} + 1 - \frac{i}{2}\sqrt{\lambda - 1}\right)}{\Gamma\left(\frac{l}{2} + \frac{1}{2} - \frac{i}{2}\sqrt{\lambda - 1}\right)} \right|^2 \frac{|w(0)|^2}{\left(e^{\frac{\pi}{2}\sqrt{\lambda - 1}} + (-1)^l e^{-\frac{\pi}{2}\sqrt{\lambda - 1}}\right)^2} \right. \\ &\quad \left. + \frac{1}{4} \left| \frac{\Gamma\left(\frac{l}{2} + \frac{1}{2} - \frac{i}{2}\sqrt{\lambda - 1}\right)}{\Gamma\left(\frac{l}{2} + 1 - \frac{i}{2}\sqrt{\lambda - 1}\right)} \right|^2 \frac{|w'(0)|^2}{\left(e^{\frac{\pi}{2}\sqrt{\lambda - 1}} - (-1)^l e^{-\frac{\pi}{2}\sqrt{\lambda - 1}}\right)^2} \right]. \end{aligned}$$

There exist $C', C'' > 0$ independent of $\lambda > 1$, $l \in \mathbb{N}$ such that if l is even

(VI.13)

$$\begin{aligned} C' &\left[\tanh\left(\frac{\pi}{2}\sqrt{\lambda - 1}\right) (l + 1 + \sqrt{\lambda - 1}) |w(0)|^2 + \coth\left(\frac{\pi}{2}\sqrt{\lambda - 1}\right) (l + 1 + \sqrt{\lambda - 1})^{-1} |w'(0)|^2 \right] \\ &\leq \sqrt{\lambda - 1} \left(|w_{in/out}^+|^2 + |w_{in/out}^-|^2 \right) \leq \\ C'' &\left[\tanh\left(\frac{\pi}{2}\sqrt{\lambda - 1}\right) (l + 1 + \sqrt{\lambda - 1}) |w(0)|^2 + \coth\left(\frac{\pi}{2}\sqrt{\lambda - 1}\right) (l + 1 + \sqrt{\lambda - 1})^{-1} |w'(0)|^2 \right], \end{aligned}$$

and if l is odd

(VI.14)

$$\begin{aligned} C' &\left[\coth\left(\frac{\pi}{2}\sqrt{\lambda - 1}\right) (l + 1 + \sqrt{\lambda - 1}) |w(0)|^2 + \tanh\left(\frac{\pi}{2}\sqrt{\lambda - 1}\right) (l + 1 + \sqrt{\lambda - 1})^{-1} |w'(0)|^2 \right] \\ &\leq \sqrt{\lambda - 1} \left(|w_{in/out}^+|^2 + |w_{in/out}^-|^2 \right) \leq \\ C'' &\left[\coth\left(\frac{\pi}{2}\sqrt{\lambda - 1}\right) (l + 1 + \sqrt{\lambda - 1}) |w(0)|^2 + \tanh\left(\frac{\pi}{2}\sqrt{\lambda - 1}\right) (l + 1 + \sqrt{\lambda - 1})^{-1} |w'(0)|^2 \right], \end{aligned}$$

Finally we have:

(VI.15)

$$|w(t)| \leq 4e^{\frac{\pi}{2}\sqrt{\lambda - 1}} \left(|A^+| + |A^-| \right).$$

(VI.16)

$$|w'(t)| \leq 8e^{\frac{\pi}{2}\sqrt{\lambda - 1}} (l + 1 + \sqrt{\lambda - 1}) \left(|A^+| + |A^-| \right).$$

We remark that (VI.13) and (VI.14) lead to a sharp estimate of the initial energy:

(VI.17)

$$E(w, 0) \leq \frac{1}{C' \tanh\left(\frac{\pi}{2}\sqrt{\lambda - 1}\right)} \left(1 + \frac{l + 1}{\sqrt{\lambda - 1}} \right) E(w, \infty).$$

We have to compare it with the usual inequality obtained by the Grönwall lemma for $t \leq 0$:

(VI.18)

$$\begin{aligned} E(w, t) &= E(w, \infty) - \int_{-\infty}^t l(l + 1) \frac{\sinh s}{\cosh^3 s} |w(s)|^2 ds \\ &\leq E(w, \infty) + \int_{-\infty}^t \frac{-2l(l + 1) \frac{\sinh s}{\cosh^3 s}}{\lambda - 1 + \frac{l(l + 1)}{\cosh^2 s}} E(w, s) ds \\ &\leq \left(1 + \frac{l(l + 1)}{(\lambda - 1) \cosh^2 t} \right) E(w, \infty). \end{aligned}$$

Parenthetically, we mention that the De Sitter propagator in flat coordinates has been studied in [31], and some energy estimates for general wave equations with time dependent coefficients are investigated in [35].

Proof. To prove (VI.15) we use the Mehler Dirichlet integral to estimate the Ferrers function for $t > 0$:

$$\begin{aligned}
\left| P_l^{i\sqrt{\lambda-1}}(\tanh t) \right| &\leq \sqrt{\frac{2}{\pi}} \frac{1}{\left| \Gamma\left(\frac{1}{2} - i\sqrt{\lambda-1}\right) \right|} \int_0^{\arccos \tanh t} \frac{1}{\sqrt{\cos s - \tanh t}} ds \\
&\leq \sqrt{2 \cosh(\pi\sqrt{\lambda-1})} \int_{\tanh t}^1 \frac{1}{\sqrt{u - \tanh t} \sqrt{1-u^2}} du \\
&\leq e^{\frac{\pi}{2}\sqrt{\lambda-1}} \int_{\tanh t}^1 \frac{1}{\sqrt{u - \tanh t} \sqrt{1-u}} du \\
&\leq e^{\frac{\pi}{2}\sqrt{\lambda-1}} \sqrt{\frac{2}{1 - \tanh t}} \left(\int_{\tanh t}^{\frac{1+\tanh t}{2}} \frac{1}{\sqrt{u - \tanh t}} du + \int_{\frac{1+\tanh t}{2}}^1 \frac{1}{\sqrt{1-u}} du \right) \\
&\leq 4e^{\frac{\pi}{2}\sqrt{\lambda-1}}.
\end{aligned}$$

Now (VI.16) follows from this estimate and the formula (14.10.5) of [43]. The asymptotics of w at $t = \pm\infty$ are deduced by some elementary but long and tedious computations from the asymptotics of the Ferrers functions (see (14.8.1), (14.9.7) in [43]). The decay of the energy follows from the sign of its time derivative and we get (VI.12) from the previous results and the reflection formula for the gamma function, $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$. To deduce (VI.13) and (VI.14) from (VI.12), we use the estimate $\Gamma(z + \frac{1}{2})/\Gamma(z) \sim z^{\frac{1}{2}}$ for $|z| \rightarrow \infty$, $|\arg(z)| \leq \pi - \delta$.

Q.E.D.

Theorem VI.3. *Let v be a solution of (VI.1) in $C^0(\mathbb{R}_t; X^1) \cap C^1(\mathbb{R}_t; X^0)$. Then there exist unique $v_{in(out)} \in C^0(\mathbb{R}_t; \dot{X}^1) \cap C^1(\mathbb{R}_t; X^0)$ solutions of (VI.3) such that*

$$(VI.19) \quad E_\infty(v - v_{in(out)}, t) \rightarrow 0, \quad t \rightarrow -(+)\infty.$$

The wave operators $v \mapsto v_{in(out)}$ are one-to-one and we have

$$(VI.20) \quad E_\infty(v_{out}) = E_\infty(v_{in}).$$

Since $(v_{in/out}, \partial_t v_{in/out})$ are almost periodic $\dot{X}^1 \times X^0$ -valued functions if $M > 0$ or if $M = 0$ and $v_{in/out} \in C^1(\mathbb{R}_t; X_\perp^0)$, we deduce that $(v, \partial_t v)$ is an asymptotically almost periodic $\dot{X}^1 \times X^0$ -valued function if its effective mass is not zero, i.e. $M > 0$ or $\partial_\psi v \neq 0$.

Proof of Theorem VI.3. Following (V.25), if $M > 0$ or if $M = 0$ and $v \in C^0(\mathbb{R}_t; X_\perp^1) \cap C^1(\mathbb{R}_t; X_\perp^0)$, we can write the field as

$$(VI.21) \quad v(t, x, \Omega_2, \psi) = \sum_{k,l,m,n,\pm} A_{k,l,m,n}^\pm P_l^{i\sqrt{\lambda_{n,k}-1}}(\pm \tanh t) w_{n,k}(\sinh x) Y_{l,m}(\Omega_2) e^{in\psi},$$

and if $M = 0$ and $v \in C^0(\mathbb{R}_t; X_0^1) \cap C^1(\mathbb{R}_t; X_0^0)$

$$(VI.22) \quad v(t, x, \Omega_2) =$$

$$\frac{2}{\pi} \lim_{A \rightarrow \infty} \left(\frac{1}{\sinh(2x)} \right)^{\frac{1}{2}} \int_1^A \sum_{l,m,\pm} A_{l,m}^\pm(\lambda) P_l^{i\sqrt{\lambda-1}}(\pm \tanh t) Y_{l,m}(\Omega_2) \tanh\left(\frac{\pi}{2}\sqrt{\lambda-1}\right) Q_{-\frac{1}{2}}^{\frac{i}{2}\sqrt{\lambda-1}}(\coth(2x)) d\lambda$$

where these expansions hold in $C^0(\mathbb{R}_t; X^1) \cap C^1(\mathbb{R}_t; X^0)$. We introduce

$$\begin{aligned} w_{in(out)}^{-(+)}(k, l, m, n) &= \frac{1}{\Gamma(1 - i\sqrt{\lambda_{n,k} - 1})} A_{k,l,m,n}^{-(+)}, \\ w_{in(out)}^{+(-)}(k, l, m, n) &= (-1)^l \frac{\Gamma(l + 1 + i\sqrt{\lambda_{n,k} - 1})}{\Gamma(l + 1 - i\sqrt{\lambda_{n,k} - 1})\Gamma(1 + i\sqrt{\lambda_{n,k} - 1})} A_{k,l,m,n}^{+(-)}, \\ w_{in(out)}^{-(+)}(l, m; \lambda) &= \frac{1}{\Gamma(1 - i\sqrt{\lambda - 1})} A_{l,m}^{-(+)}(\lambda), \\ w_{in(out)}^{+(-)}(l, m; \lambda) &= (-1)^l \frac{\Gamma(l + 1 + i\sqrt{\lambda - 1})}{\Gamma(l + 1 - i\sqrt{\lambda - 1})\Gamma(1 + i\sqrt{\lambda - 1})} A_{l,m}^{+(-)}(\lambda), \end{aligned}$$

and we put

$$(VI.23) \quad v_{in(out)}(t, x, \Omega_2, \psi) = \sum_{k,l,m,n,\pm} w_{in(out)}^{\pm}(k, l, m, n) e^{\pm it\sqrt{\lambda_{n,k}-1}} w_{n,k}(\sinh x) Y_{l,m}(\Omega_2) e^{in\psi},$$

$$(VI.24)$$

$$v_{in(out)}(t, x, \Omega_2) =$$

$$\frac{2}{\pi} \lim_{A \rightarrow \infty} \left(\frac{1}{\sinh(2x)} \right)^{\frac{1}{2}} \int_1^A \sum_{l,m,\pm} w_{in(out)}^{\pm}(l, m; \lambda) e^{\pm it\sqrt{\lambda-1}} Y_{l,m}(\Omega_2) \tanh\left(\frac{\pi}{2}\sqrt{\lambda-1}\right) Q_{-\frac{1}{2}}^{\frac{i}{2}\sqrt{\lambda-1}}(\coth(2x)) d\lambda.$$

Thanks to the decay of the energy, these series are converging in $C^0(\mathbb{R}_t; \dot{X}^1) \cap C^1(\mathbb{R}_t; X^0)$ and the theorem follows from lemma VI.2.

Q.E.D.

We remark that the energy of the asymptotic states is given if $M > 0$ or $v \in C^0(\mathbb{R}_t; X_{\perp}^0)$ by

$$\begin{aligned} (VI.25) \quad E_{\infty}(v_{in(out)}) &= 2 \sum_{k,l,m,n,\pm} (\lambda_{n,k} - 1) |w_{in(out)}^{\pm}(k, l, m, n)|^2 \\ &= \frac{2}{\pi} \sum_{k,l,m,n,\pm} (\lambda_{n,k} - 1)^{\frac{1}{2}} \sinh\left(\pi\sqrt{\lambda_{n,k} - 1}\right) |A_{k,l,m,n}^{\pm}|^2, \end{aligned}$$

and if $M = 0$ and $v \in C^0(\mathbb{R}_t; X_0^0)$ by

$$\begin{aligned} (VI.26) \quad E_{\infty}(v_{in(out)}) &= 2 \sum_{l,m,\pm} \int_1^{\infty} (\lambda - 1) |w_{in(out)}^{\pm}(l, m; \lambda)|^2 d\lambda \\ &= \frac{2}{\pi} \sum_{l,m,\pm} \int_1^{\infty} (\lambda - 1)^{\frac{1}{2}} \sinh\left(\pi\sqrt{\lambda - 1}\right) |A_{l,m}^{\pm}(\lambda)|^2 d\lambda. \end{aligned}$$

We end this part by investigating the massless case, $M = 0$, $\partial_{\psi} v = 0$. We introduce the subspace \dot{X}_0^1 that the closure of $\{v \in C_0^{\infty}(\Sigma), \partial_{\psi} v = 0\}$ for the norm

$$\|v\|_{\dot{X}^1}^2 = \int_{\Sigma} |\partial_x v|^2 d\mu.$$

We put

$$(VI.27) \quad v(x, \Omega_2) = \frac{w(x, \Omega_2)}{\sqrt{\sinh(2x)}}.$$

Then

$$(VI.28) \quad 2\|v\|_{\dot{X}^1}^2 = \int_{]0,\infty[\times S^2} |\partial_x w|^2 + \left(1 - \frac{1}{\sinh^2(2x)}\right) |w|^2 dx d\Omega_2$$

and taking advantage of the Hardy inequality, we introduce the space K^1 closure of $C_0^\infty([0, \infty[\times S^2)$ for the norm

$$(VI.29) \quad \|w\|_{K^1}^2 := \int_{[0, \infty[\times S^2} |\partial_x w|^2 - \frac{1}{\sinh^2(2x)} |w|^2 dx d\Omega$$

Since $v_{in(out)}$ satisfy (VI.3), we have

$$(VI.30) \quad \left[\partial_t^2 - \partial_x^2 - \frac{1}{\sinh^2(2x)} \right] w_{in(out)} = 0, \quad v_{in(out)}(t, x, \Omega_2) = \frac{w_{in(out)}(t, x, \Omega_2)}{\sqrt{\sinh(2x)}}.$$

We have seen in the proof of Proposition V.4 that $\left(-\partial_x^2 - \frac{1}{\sinh^2(2x)} + 1\right)^{-1} - \left(-\partial_x^2 - \frac{1}{4x^2} + 1\right)^{-1}$ is compact on $L^2(0, \infty)$ if the operators are endowed with the domain (V.38). Hence we can apply the technics used in [4] to compare the dynamics of (VI.30) with

$$(VI.31) \quad \left[\partial_t^2 - \partial_x^2 - \frac{1}{4x^2} \right] \hat{w}_{in(out)} = 0 \text{ on } (0, \infty)_x \times S^2,$$

and we can prove that there exists a unique $\hat{w}_{in(out)} \in C^0(\mathbb{R}_t; K^1)$ with $\partial_t \hat{w}_{in(out)} \in C^0(\mathbb{R}_t; L^2([0, \infty[\times S^2))$ satisfying

$$(VI.32) \quad \|\hat{w}_{in(out)}(t) - w_{in(out)}(t)\|_{K^1} + \|\partial_t \hat{w}_{in(out)}(t) - \partial_t w_{in(out)}(t)\|_{L^2} \rightarrow 0, \quad t \rightarrow -(+)\infty.$$

We put for $\mathbf{x} \in \mathbb{R}^2$

$$(VI.33) \quad W_{in(out)}(t, \mathbf{x}) := \frac{\hat{w}_{in(out)}(t, |\mathbf{x}|)}{|\mathbf{x}|^{\frac{1}{2}}}$$

and we note that

$$(VI.34) \quad \partial_t^2 W_{in(out)} - \Delta_{\mathbb{R}^2} W_{in(out)} = 0 \text{ in } \mathbb{R}_t \times \mathbb{R}_{\mathbf{x}}^2 \times S^2$$

that is finite energy associated with the following conserved currents

$$(VI.35) \quad \begin{aligned} \int_{S^2} \int_{\mathbb{R}_{\mathbf{x}}^2} \left[|\partial_t W_{in(out)}|^2 + |\nabla_{\mathbf{x}} W_{in(out)}|^2 \right] d\mathbf{x} d\Omega_2 \\ = \int_{S^2} \int_{[0, \infty[_x} \left[|\partial_t \hat{w}_{in(out)}|^2 + |\partial_x \hat{w}_{in(out)}|^2 + \frac{1}{4x^2} |\hat{w}_{in(out)}|^2 \right] dx d\Omega_2 \\ = \int_{S^2} \int_{[0, \infty[_x} \left[|\partial_t w_{in(out)}|^2 + |\partial_x w_{in(out)}|^2 + \frac{1}{\sinh^2(2x)} |w_{in(out)}|^2 \right] dx d\Omega_2 \end{aligned}$$

We conclude that the solutions $u \in C^0(\mathbb{R}_t; W_0^1) \cap C^1(\mathbb{R}_t; X_0^0)$ of the massless equation (IV.1) with $M = 0$, are asymptotic with free waves in the sense of the previous energy:

$$(VI.36) \quad \cosh(t) \sqrt{\frac{\sinh(2|\mathbf{x}|)}{|\mathbf{x}|}} u(t, x = |\mathbf{x}|, \Omega_2) \sim W_{in(out)}(t, \mathbf{x}, \Omega_2), \quad t \rightarrow -(+)\infty,$$

hence they are dispersive waves.

VII. HAWKING WORMHOLE

In [33], Hawking introduced an Euclidean wormhole. The Lorentzian version of this wormhole is the submanifold $\{\psi = 0, \pi\}$ (or $z = 0$) of the Witten space time \mathcal{M} , and we consider it as a *sui generis* 1 + 3 dimensional, globally hyperbolic manifold \mathcal{W} . In this section we present its main geometrical properties. If we use the $(t, y, \Omega_2) \in \mathbb{R} \times \mathbb{R} \times S^2$ coordinates, the metric on \mathcal{W} is:

$$(VII.1) \quad ds_{\mathcal{W}}^2 = \rho^2 dt^2 - \frac{(1 + \rho)^2}{\rho^2} e^{-2\rho} dy^2 - \rho^2 \cosh^2 t d\Omega_2^2, \quad \rho = \frac{1}{2} W \begin{pmatrix} +2 \\ -2, y^2 \end{pmatrix}.$$

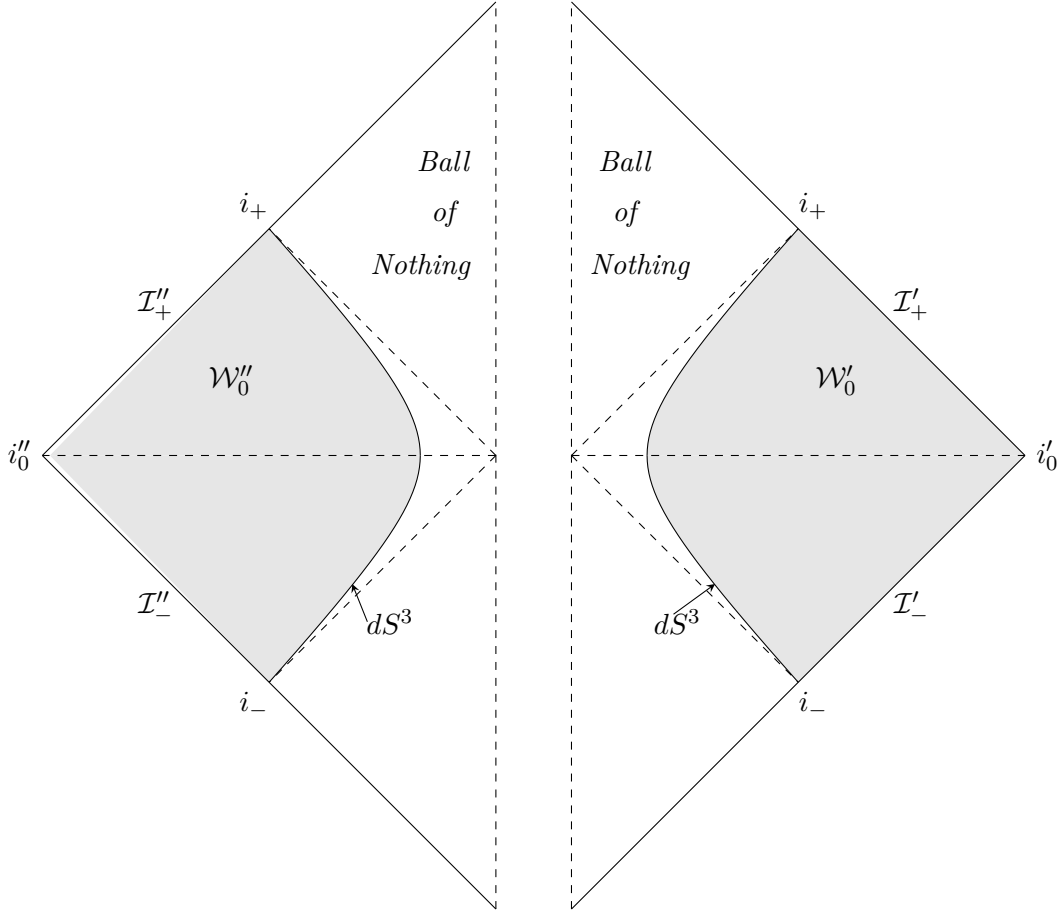


FIGURE 3. Penrose Conformal Diagram of the Hawking Wormhole. A ball of nothing is removed from two copies of the Minkowski spacetime. The two De Sitter boundaries dS^3 are identified to form the throat of the wormhole.

Using the coordinates $r = \frac{y}{|y|} \sqrt{\left(\frac{1}{2}W\left(\begin{smallmatrix} +2 \\ -2 \end{smallmatrix}, y^2\right)\right)^2 - 1}$ solution of the transcendental equation $y = \frac{re^{\sqrt{r^2+1}}}{1+\sqrt{r^2+1}}$, and $x := \text{arcsinh}(r)$, two other descriptions are

$$(VII.2) \quad \mathcal{W} = \mathbb{R}_t \times \mathbb{R}_r \times S^2, \quad ds_{\mathcal{W}}^2 = (r^2 + 1)dt^2 - dr^2 - (r^2 + 1)\cosh^2 t d\Omega_2^2$$

$$(VII.3) \quad \mathcal{W} = \mathbb{R}_t \times \mathbb{R}_x \times S^2, \quad ds_{\mathcal{W}}^2 = \cosh^2(x) \left[dt^2 - dx^2 - \cosh^2 t d\Omega_2^2 \right].$$

Its equatorial section $\theta = \frac{\pi}{2}$, at fixed t , is depicted by Figure 2. We note that with the (τ, ξ) coordinates (II.3), we have

$$ds_{\mathcal{W}}^2 \sim d\tau^2 - d\xi^2 - \xi^2 d\Omega_2^2, \quad y \rightarrow \pm\infty.$$

Therefore \mathcal{W} is a dynamic spheric wormhole because, (1) it connects two asymptotically flat spacetimes, (2) it has a dynamic throat, *i.e.* a time dependent surface of minimal area, connecting both these asymptotically flat universes. This throat is just the bubble of nothing $y = 0$, or $\rho = 1$, that is the 2+1 de Sitter space-time (II.15). Another nice description of the wormhole consists in using the (T, Σ) coordinates (II.22). The expression (II.23) shows that \mathcal{W} can be represented as

$\{(T, \Sigma) \in \mathbb{R}^2, \Sigma^2 - T^2 \geq \frac{1}{4}\} \times S^2$ endowed with the conformally flat metric

$$(VII.4) \quad ds_{\mathcal{W}}^2 = \left(1 + \frac{1}{4(\Sigma^2 - T^2)}\right)^2 \{dT^2 - d\Sigma^2 - \Sigma^2 d\Omega_2^2\},$$

where the points (T, Σ, Ω_2) and $(T, -\Sigma, \Omega_2)$ are identified if $\Sigma^2 - T^2 = \frac{1}{4}$. We compare the Hawking wormhole with a dynamic spheric wormhole \mathcal{W}_0 , built by Minkowski surgery as follows. We take two copies of the Minkowski space-time $\mathbb{R}_T \times \mathbb{R}_X^3$. We remove the subset $|X|^2 \leq T^2 + \frac{1}{4}$ (“*ball of nothing*”) from each spacetime and so we get two pieces \mathcal{W}'_0 and \mathcal{W}''_0 . We identify at the boundaries $|X|^2 = T^2 + \frac{1}{4}$ that are just dS^3 . Outside this contracting-expanding throat, we endow \mathcal{W}_0 with the flat metric

$$ds_{\mathcal{W}_0}^2 = dT^2 - dX^2.$$

Writing in spherical coordinates $X = |\Sigma| \Omega_2 \in \mathbb{R}^3$, we obtain

$$(VII.5) \quad ds_{\mathcal{W}}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu := \left(1 + \frac{1}{4(|X|^2 - T^2)}\right)^2 \{dT^2 - dX^2\}$$

that is just the Lorentzian version of the Euclidean wormhole proposed by Hawking in [33]. We conclude that the Hawking wormhole and \mathcal{W}_0 are conformal equivalent.

Now we compute the Einstein tensor and constat that the weakest condition of energy, the null one, is violated. From (VII.5) it is easy to obtain the Ricci curvature tensor, the Ricci scalar and the energy momentum tensor $\bar{T}_{\mu\nu} := \frac{1}{2}\bar{R}\bar{g}_{\mu\nu} - \bar{R}_{\mu\nu}$ for the metric (VII.5), by using the general relations concerning the conformal equivalent metrics, $\bar{g}_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}(x)$:

$$\bar{R}^\nu_\mu = \Omega^{-2}R^\nu_\mu - (n-2)\Omega^{-1}(\Omega^{-1})_{;\mu\rho}g^{\rho\nu} + (n-2)^{-1}\Omega^{-n}(\Omega^{n-2})_{;\rho\sigma}g^{\rho\sigma}\delta^\nu_\mu,$$

$$\bar{R} = \Omega^{-2}R + 2(n-1)\Omega^{-3}\Omega_{;\mu\nu}g^{\mu\nu} + (n-1)(n-4)\Omega^{-4}\Omega_{;\mu}\Omega_{;\nu}g^{\mu\nu}.$$

Taking $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and $\Omega = 1 + \frac{1}{4}(|X|^2 - T^2)^{-1}$, we calculate these quantities and in particular we get that the Ricci scalar of the Hawking wormhole is zero:

$$(VII.6) \quad \bar{R} = 0,$$

and writing $T = X^0$, the Ricci tensor is for $1 \leq j \neq k \leq 3$:

$$\begin{aligned} \bar{R}_{00} &= \frac{|X|^2 + 3T^2}{\left(|X|^2 - T^2 + \frac{1}{4}\right)^2 (|X|^2 - T^2)}, \quad \bar{R}_{0j} = \frac{-4TX^j}{\left(|X|^2 - T^2 + \frac{1}{4}\right)^2 (|X|^2 - T^2)}, \\ \bar{R}_{jj} &= \frac{4(X^j)^2 - |X|^2 + T^2}{\left(|X|^2 - T^2 + \frac{1}{4}\right)^2 (|X|^2 - T^2)}, \quad \bar{R}_{jk} = \frac{4X^j X^k}{\left(|X|^2 - T^2 + \frac{1}{4}\right)^2 (|X|^2 - T^2)}. \end{aligned}$$

We get that the Null Energy Condition is violated since for the null vector field $V = \partial_0 \pm \partial_j$ we have

$$(VII.7) \quad \bar{T}_{\mu\nu} V^\mu V^\nu = \frac{-4(T \mp X^j)^2}{\left(|X|^2 - T^2 + \frac{1}{4}\right)^2 (|X|^2 - T^2)}.$$

Hence, this manifold shares the common default of the Lorentzian wormholes: to exist, it should involve some exotic matter (see [51] for a general discussion on the wormholes).

As regards its traversability, Proposition III.1 assures that $(t, y = t, \omega_0)$ is a null geodesic and for all the time-like geodesics, y is a periodic function of its proper time. We conclude that this wormhole is weakly traversable in the sense that the light ray can cross the throat and go from $y = \pm\infty$ to $y = \mp\infty$, but unfortunately, any massive inertial observer is condemned to stay in the vicinity of the contracting-expanding throat, to oscillate around it forever.

VIII. WAVES IN THE HAWKING WORMHOLE

We investigate the solutions of the Klein-Gordon equation on the wormhole that has the following form in (t, x, Ω_2) coordinates:

$$(VIII.1) \quad \left[\frac{1}{\cosh^2 t} \partial_t (\cosh^2 t \partial_t) - \frac{1}{\cosh^2 x} \partial_x (\cosh^2 x \partial_x) - \frac{1}{\cosh^2 t} \Delta_{S^2} + M^2 \cosh^2 x \right] u = 0.$$

Due to the damping at the infinity, it is convenient to work with the profile of the field, $v(t, x, \Omega_2) := \cosh(t) \cosh(x) u(t, x, \Omega_2)$, that satisfies the equation

$$(VIII.2) \quad \left[\partial_t^2 - \partial_x^2 - \frac{1}{\cosh^2 t} \Delta_{S^2} + M^2 \cosh^2 x \right] v = 0.$$

The natural energy associated with v is

$$(VIII.3) \quad E(v, t) := \int_{-\infty}^{\infty} \int_{S^2} |\partial_t v(t)|^2 + |\partial_x v(t)|^2 + \frac{1}{\cosh^2 t} |\nabla_{S^2} v(t)|^2 + M^2 \cosh^2(x) |v(t)|^2 dx d\Omega_2,$$

hence we introduce the space $H^{1,M}(\mathbb{R} \times S^2)$ defined as the closure of $C_0^\infty(\mathbb{R} \times S^2)$ for the norm

$$(VIII.4) \quad \|v\|_{H^{1,M}}^2 := \int_{-\infty}^{\infty} \int_{S^2} |\partial_x v|^2 + |\nabla_{S^2} v|^2 + M^2 \cosh^2(x) |v|^2 dx d\Omega_2.$$

We can easily see that if $M > 0$, $H^{1,M}(\mathbb{R} \times S^2)$ is compactly embedded in $L^2(\mathbb{R} \times S^2)$. In contrast we warn that if the mass is zero, $H^{1,0}(\mathbb{R} \times S^2)$ is not a subspace of distributions. To prove this remark, we take $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi(x, \Omega_2) = 1$ for $|x| \leq 1$. Then $\varphi_n(x, \Omega_2) := \varphi(x/n)$ tends to zero in $H^{1,0}(\mathbb{R} \times S^2)$ as $n \rightarrow \infty$, and to 1 in $\mathcal{D}'(\mathbb{R} \times S^2)$. Nevertheless this pathology is limited to the component on the rotationnally invariant subspace $\dot{H}^1(\mathbb{R}) \otimes 1$ of which the orthogonal in $H^{1,0}$ is a subspace of $H^1(\mathbb{R} \times S^2)$. For this component, the equation (VIII.2) is reduced to the trivial wave equation $\partial_t^2 v - \partial_x^2 v = 0$ on $\mathbb{R} \times S^2$ and the PDE is satisfied in a spectral sense by the solution $v(t, x, \Omega_2) = v^+(x+t) + v^-(x-t)$.

If $M > 0$ we introduce the modified Mathieu operator

$$L_M := -\frac{d^2}{dx^2} + M^2 \cosh^2 x, \quad D(L_M) := \left\{ v \in L^2(\mathbb{R}); L_M v \in L^2(\mathbb{R}) \right\}.$$

It is well known that L_M is a densely defined positive selfadjoint operator on $L^2(\mathbb{R})$. We denote $0 < \lambda_k \leq \lambda_{k+1} \leq \dots$, $k \in \mathbb{N}$, the sequence of its eigenvalues, and $w_k(x)$ a Hilbert basis of eigenfunctions, with $L_M w_k = \lambda_k w_k$.

Theorem VIII.1. *For $M \geq 0$, given $v_0 \in H^{1,M}(\mathbb{R} \times S^2)$, $v_1 \in L^2(\mathbb{R} \times S^2)$, there exists a unique $v \in C^0(\mathbb{R}_t; H^{1,M})$ with $\partial_t v \in C^0(\mathbb{R}_t; L^2(\mathbb{R} \times S^2))$, solution of (VIII.2) such that $v(0) = v_0$, $\partial_t v(0) = v_1$. Its energy $E(v, t)$ is decreasing with $|t|$. If $M > 0$, v can be written as*

$$(VIII.5) \quad v(t, x, \Omega_2) = \sum_{k,l,m,\pm} A_{k,l,m}^\pm w_k(x) Y_{l,m}(\Omega_2) P_l^{i\sqrt{\lambda_k}}(\pm \tanh t),$$

where the series is converging in $C^0(\mathbb{R}_t; H^{1,M}) \cap C^1(\mathbb{R}_t; L^2)$. If $M = 0$, v can be written as

$$(VIII.6) \quad v(t, x, \Omega_2) = \frac{1}{2\pi} \sum_{l,m,\pm} Y_{l,m}(\Omega_2) \lim_{A \rightarrow \infty} \int_{-A}^A e^{ix\xi} P_l^{i|\xi|}(\pm \tanh t) A_{l,m}^\pm(\xi) d\xi,$$

where the series is converging in $C^0(\mathbb{R}_t; H^{1,0}) \cap C^1(\mathbb{R}_t; L^2)$.

Proof. Since the wormhole is globally hyperbolic, the Cauchy problem is well posed in C_0^∞ hence on $H^{1,M} \times L^2$ by density thanks to the energy estimate (see also [20]). We prove the expansion of the solution using the same method as for the Witten space.

Q.E.D.

We make some remarks on the massless case $M = 0$ for which the D'Alembertian is conformal invariant since the Ricci scalar is zero. Therefore if $\bar{g} = \Omega^2 g$ then $\bar{\square} \bar{u} = 0$ iff $\square u = 0$ with $\bar{u} = \Omega^{-1} u$. We introduce for $|X|^2 - T^2 \geq 1/4$

$$(VIII.7) \quad u_{\pm}(T, X) = \left(1 + e^{-2|x|}\right) u(t, x, \Omega_2), \quad \pm x \geq 0.$$

Then u is a smooth solution of (VIII.1) iff u_{\pm} are solutions of the flat D'Alembertian

$$\partial_T^2 u_{\pm} - \Delta_{\mathbb{R}_X^3} u_{\pm} = 0 \quad \text{in } |X|^2 - T^2 > 1/4$$

and

$$u_+ = u_-, \quad T \partial_T u_+ + X \cdot \nabla_X u_+ = -T \partial_T u_- - X \cdot \nabla_X u_- \quad \text{if } |X|^2 - T^2 = 1/4.$$

This system can be decoupled by introducing free waves satisfying the Dirichlet or the Neumann condition on the boundary. We define

$$(VIII.8) \quad u_D(T, X) := u_+ - u_-, \quad u_N := u_+ + u_-,$$

Then u is solution of (VIII.1) iff u_D and u_N are solutions of

$$(VIII.9) \quad \left(\partial_T^2 - \Delta_X\right) u_{D,N} = 0 \quad \text{in } |X|^2 - T^2 > 1/4, \quad u_D = 0, \quad (T \partial_T + X \cdot \nabla_X) u_N = 0 \quad \text{on } |X|^2 - T^2 = 1/4.$$

It will be natural to compare as $t \rightarrow \pm\infty$ the solutions of (VIII.2) with the free-field solutions of

$$(VIII.10) \quad \left[\partial_t^2 - \partial_x^2 + M^2 \cosh^2 x\right] v = 0, \quad \text{on } \mathbb{R}_t \times \mathbb{R}_x \times S^2.$$

We introduce $\dot{H}^{1,M}(\mathbb{R} \times S^2)$ defined as the closure of $C_0^\infty(\mathbb{R} \times S^2)$ for the norm

$$(VIII.11) \quad \|v\|_{\dot{H}^{1,M}}^2 := \int_{-\infty}^{\infty} \int_{S^2} |\partial_x v|^2 + M^2 \cosh^2(x) |v|^2 dx d\Omega_2.$$

As previous $\dot{H}^{1,0}(\mathbb{R} \times S^2)$ is not a subspace of distributions; it was used in [4] to study the Klein-Gordon field near the horizon of the Schwarzschild black-hole. It is easy to prove by the usual way that the Cauchy problem for (VIII.10) is well posed in $\dot{H}^{1,M} \times L^2$ and there exists a conserved energy:

$$(VIII.12) \quad E_0(v) := \int_{-\infty}^{\infty} \int_{S^2} |\partial_t v(t)|^2 + |\partial_x v(t)|^2 + M^2 \cosh^2(x) |v(t)|^2 dx d\Omega_2.$$

If $M > 0$ we have

$$(VIII.13) \quad v(t, x, \Omega_2) = \sum_{k,l,m,\pm} A_{k,l,m}^{\pm} w_k(x) Y_{l,m}(\Omega_2) e^{\pm it \sqrt{\lambda_k}}$$

where the series is converging in $C^0(\mathbb{R}_t; \dot{H}^{1,M}) \cap C^1(\mathbb{R}_t; L^2)$.

Theorem VIII.2. *For any finite energy solution v of (VIII.2), there exist unique $v_{in(out)} \in C^0(\mathbb{R}_t; \dot{H}^{1,M})$ with $\partial_t v_{in(out)} \in C^0(\mathbb{R}_t; L^2(\mathbb{R} \times S^2))$ solutions of (VIII.10) such that*

$$(VIII.14) \quad \|v(t) - v_{in(out)}(t)\|_{\dot{H}^{1,M}} + \|\partial_t v(t) - \partial_t v_{in(out)}(t)\|_{L^2(\mathbb{R} \times S^2)} \longrightarrow 0, \quad t \rightarrow -(+)\infty.$$

Moreover the wave operators $v \mapsto v_{in(out)}$ are one-to one and continuous:

$$(VIII.15) \quad E_0(v_{in(out)}) \leq E(v, 0),$$

and the scattering operator $v_{in} \mapsto v_{out}$ is isometric:

$$(VIII.16) \quad E_0(v_{in}) = E_0(v_{out}).$$

Proof. We use the expansions of theorem VIII.1 and we introduce

$$\begin{aligned} w_{in(out),}^{-(+)}(k, l, m) &= \frac{1}{\Gamma(1 - i\sqrt{\lambda_k})} A_{k,l,m}^{-(+)}, \\ w_{in(out)}^{+(-)}(k, l, m) &= (-1)^l \frac{\Gamma(l+1 + i\sqrt{\lambda_k})}{\Gamma(l+1 - i\sqrt{\lambda_k})\Gamma(1 + i\sqrt{\lambda_k})} A_{k,l,m}^{+(-)}, \\ w_{in(out)}^{-(+)}(l, m; \xi) &= \frac{1}{\Gamma(1 - i|\xi|)} A_{l,m}^{-(+)}(\xi), \\ w_{in(out)}^{+(-)}(l, m; \xi) &= (-1)^l \frac{\Gamma(l+1 + i|\xi|)}{\Gamma(l+1 - i|\xi|)\Gamma(1 + i|\xi|)} A_{l,m}^{+(-)}(\xi). \end{aligned}$$

We put

$$(VIII.17) \quad (M > 0) \quad v_{in(out)}(t, x, \Omega_2) = \sum_{k,l,m,\pm} w_{in(out),}^{\pm}(k, l, m) e^{\pm it\sqrt{\lambda_k}} w_k(x) Y_{l,m}(\Omega_2),$$

$$(VIII.18) \quad (M = 0) \quad v_{in(out)}(t, x, \Omega_2) = \frac{1}{2\pi} \sum_{l,m,\pm} Y_{l,m}(\Omega_2) \lim_{A \rightarrow \infty} \int_{-A}^A w_{in(out)}^{\pm}(l, m; \xi) e^{i(x\xi \pm t|\xi|)} d\xi.$$

Thanks to the decay of the energy, these series are converging in $C^0(\mathbb{R}_t; H^{1,M}) \cap C^1(\mathbb{R}_t; L^2(\mathbb{R} \times S^2))$ and we have

$$\begin{aligned} (VIII.19) \quad E_{\infty}(v_{in(out)}) &= 2 \sum_{k,l,m,\pm} \lambda_k |w_{in(out),}^{\pm}(k, l, m)|^2 \\ &= \frac{2}{\pi} \sum_{k,l,m,\pm} \lambda_k^{\frac{1}{2}} \sinh(\pi\sqrt{\lambda_k}) |A_{k,l,m}^{\pm}|^2, \end{aligned}$$

and if $M = 0$ by

$$\begin{aligned} (VIII.20) \quad E_{\infty}(v_{in(out)}) &= 2 \sum_{l,m,\pm} \int_{-\infty}^{\infty} |\xi|^2 |w_{in(out),}^{\pm}(l, m; \xi)|^2 d\xi \\ &= \frac{1}{\pi} \sum_{l,m,\pm} \int_{-\infty}^{\infty} |\xi| \sinh(\pi|\xi|) |A_{l,m}^{\pm}(\xi)|^2 d\xi. \end{aligned}$$

The theorem follows from lemma VI.2.

Q.E.D.

We remark that in the massive case, the fields are asymptotically almost periodic. In contrast, the massless fields are asymptotically free waves propagating in the two sheets of the wormhole, $x > 0$ and $x < 0$, since if $M = 0$ we have

$$v_{in(out)}(t, x, \Omega_2) = v_{in(out)}^{+}(t+x, \Omega_2) + v_{in(out)}^{-}(x-t, \Omega_2), \quad v_{in(out)}^{\pm} \in \dot{H}^{1,0},$$

with

$$(VIII.21) \quad v_{in(out)}^{\epsilon}(x, \Omega_2) = \frac{1}{2\pi} \sum_{l,m,\pm} Y_{l,m}(\Omega_2) \lim_{A \rightarrow \infty} \int_{-A}^A w_{in(out)}^{\pm}(l, m; \xi) \mathbf{1}_{(0,\infty)}(\pm\epsilon\xi) e^{ix\xi} d\xi, \quad \epsilon = +, -,$$

therefore we conclude that the massless fields are dispersive waves. We note that the wormhole is traversable by the massless fields satisfying for all l, m, ξ ,

$$(VIII.22) \quad w_{in}^{\pm}(l, m; \xi) \mathbf{1}_{(0,\infty)}(\pm\xi) = 0, \quad \text{or} \quad w_{in}^{\pm}(l, m; \xi) \mathbf{1}_{(0,\infty)}(\pm\xi) = 0,$$

and since

$$(VIII.23) \quad w_{out}^{\pm}(l, m, \xi) = (-1)^l \frac{\Gamma(1 \pm i|\xi|)}{\Gamma(1 \mp i|\xi|)} \frac{\Gamma(l+1 \mp i|\xi|)}{\Gamma(l+1 \pm i|\xi|)} w_{in}^{\pm}(l, m, \xi),$$

the constraint (VIII.22) is equivalent to

$$(VIII.24) \quad w_{out}^{\pm}(l, m; \xi) \mathbf{1}_{(0, \infty)}(\pm \xi) = 0, \text{ or } w_{out}^{\pm}(l, m; \xi) \mathbf{1}_{(0, \infty)}(\pm \xi) = 0$$

for all l, m, ξ . In the next part, we show that such fields exist.

IX. CLASSICAL AND QUANTUM SCATTERING

We have seen that the Witten spacetime can be defined by $t \in \mathbb{R}$, $x \geq 0$, $\Omega_d \in S^d$ and

$$ds_{Witten}^2 = \cosh^2 x \left[dt^2 - dx^2 - \cosh^2 t d\Omega_2^2 - \frac{\sinh^2 x}{\cosh^4 x} d\Omega_1^2 \right],$$

and also by $T, \Sigma \in \mathbb{R}$, $\Sigma^2 - T^2 \geq \frac{1}{4}$, $\Sigma \geq \frac{1}{2}$ and

$$ds_{Witten}^2 = \left(1 + \frac{1}{4(\Sigma^2 - T^2)} \right)^2 \left\{ dT^2 - d\Sigma^2 - \Sigma^2 d\Omega_2^2 - 16(\Sigma^2 - T^2)^2 \frac{[4(\Sigma^2 - T^2) - 1]^2}{[4(\Sigma^2 - T^2) + 1]^4} d\Omega_1^2 \right\}.$$

Here (t, x) and (T, Σ) are linked by $T = \frac{1}{2}e^x \sinh t$, $\Sigma = \frac{1}{2}e^x \cosh t$. The bubble of nothing \mathcal{B} (that is not a boundary) is the submanifold $x = 0$ or $\Sigma^2 - T^2 = \frac{1}{4}$. \mathcal{B} is just the De Sitter spacetime dS^3 . The submanifold $\{x = Cst. > 0\}$ is conformal to $dS^3 \times S^1$. The (T, Σ) coordinates allow to depict the Penrose conformal diagram of the Witten spacetime in Figure 4. We distinguish several infinities: the timelike infinities i_{\pm} that are the final points of the De Sitter submanifolds $\{x = Cst. > 0\}$ as $t \rightarrow \pm\infty$, and the null infinities \mathcal{I}_{\pm} that are the final points of the rays $x = \pm t + Cst.$, $t \rightarrow \pm\infty$. The situation for the Hawking wormhole is similar (see Figure 3). In this part we construct the classical and quantum scattering operators that associate the profile of the field near i_+ or \mathcal{I}_+ to the profile of the field near i_- or \mathcal{I}_- . In the previous sections we have investigated the asymptotic behaviours of the profiles of the Klein-Gordon fields in the Witten or Hawking spacetimes, that are solutions of

$$(IX.1) \quad \left[\partial_t^2 - \frac{1}{\cosh^2 t} \Delta_{S^2} + A \right] v = 0$$

by comparing them with the solutions $v_{in(out)}$ of the asymptotic equation

$$(IX.2) \quad \left[\partial_t^2 + A \right] v = 0.$$

Here A is a self-adjoint differential operator given for the Witten spacetime by

$$(IX.3) \quad A = -\frac{1}{\sinh(2x)} \partial_x (\sinh(2x) \partial_x) - \frac{\cosh^4 x}{\sinh^2 x} \partial_{\psi}^2 + M^2 \cosh^2 x - 1, \quad 0 < x, \quad \psi \in S^1.$$

and for the Hawking wormhole A is the modified Mathieu operator

$$(IX.4) \quad A = -\frac{\partial^2}{\partial x^2} + M^2 \cosh^2 x, \quad x \in \mathbb{R}.$$

In the massless case ($M = 0$ and $\partial_{\psi} u = 0$), the profiles of the waves are dispersive, and the asymptotic equation (IX.2) describes the behaviour of the fields near the null infinities \mathcal{I}_{\pm} . In this case the situation is similar to the scattering problem for a perturbation of the Minkowski metric. In contrast, in the massive case, the profiles are asymptotically quasi-periodic and the asymptotic equation (IX.2) describes the behaviour of the fields near the timelike infinities i_{\pm} . In this case the situation is similar to the scattering problem in the De Sitter spacetimes in global coordinates (for general results on the asymptotic behaviours of the waves in De Sitter-like universes, see [49]).

We now present our functional framework. We have constructed the wave operators

$$(IX.5) \quad \Omega_{in(out)} : (v(0, \cdot), \partial_t v(0, \cdot)) \mapsto (v_{in(out)}(0, \cdot), \partial_t v_{in(out)}(0, \cdot))$$

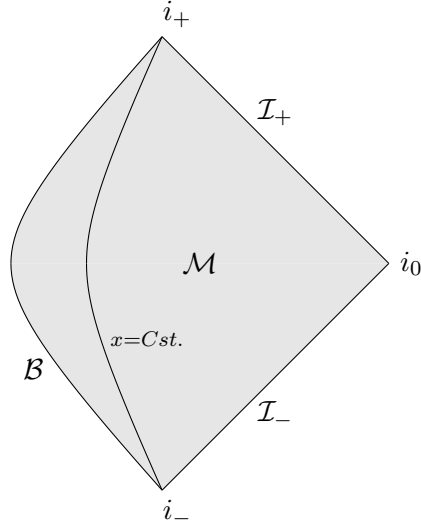


FIGURE 4. Penrose Conformal Diagram of the Witten spacetime.

defined on the Hilbert spaces (of Sobolev type $H^1 \times L^2$) associated with the energy for (IX.1),

$$\|\partial_t u\|_{X^0}^2 + \|[-\Delta_{S^2} \otimes 1 + 1 \otimes A]^{\frac{1}{2}} u\|_{X^0}^2,$$

and for (IX.2)

$$\|\partial_t u\|_{X^0}^2 + \|[1 \otimes A]^{\frac{1}{2}} u\|_{X^0}^2.$$

We now extend these operators and construct their inverses and also the scattering operator

$$(IX.6) \quad S : (v_{in}(0, \cdot), \partial_t v_{in}(0, \cdot)) \mapsto (v_{out}(0, \cdot), \partial_t v_{out}(0, \cdot))$$

on the Hilbert spaces (of Sobolev type $H^{\frac{1}{2}} \times H^{-\frac{1}{2}}$) associated with the norms

$$\|[-\Delta_{S^2} \otimes 1 + 1 \otimes A]^{-\frac{1}{4}} \partial_t u\|_{X^0}^2 + \|[-\Delta_{S^2} \otimes 1 + 1 \otimes A]^{\frac{1}{4}} u\|_{X^0}^2,$$

$$\|[1 \otimes A]^{-\frac{1}{4}} \partial_t u\|_{X^0}^2 + \|[1 \otimes A]^{\frac{1}{4}} u\|_{X^0}^2.$$

These spaces, together with the existence of a propagator and the symplectic form (IX.22) below, completely define the one-particle structure suitable for the second quantization of the fields (see *e.g.* [24], [26], [30], [38]). It will be convenient to express these quantities by using the spectral expansions. We begin by the massive case in the Witten space-time. If $M > 0$, we define for v written as

$$v(x, \Omega_2, \psi) = \sum_{k,l,m,n} v_{k,l,m,n} w_{n,k}(\sinh x) Y_{l,m}(\Omega_2) e^{in\psi}, \quad v_{k,l,m,n} \in \mathbb{C},$$

$$(IX.7) \quad \|v\|_{X_M^{\pm\frac{1}{2}}}^2 := \sum_{k,l,m,n} \left(l+1 + \sqrt{\lambda_{n,k}-1} \right)^{\pm 1} |v_{k,l,m,n}|^2, \quad X_M^{\pm\frac{1}{2}} := \left\{ v, \|v\|_{X_M^{\pm\frac{1}{2}}} < \infty \right\},$$

$$(IX.8) \quad \|v\|_{\Sigma_M^{\pm\frac{1}{2}}}^2 := \sum_{k,l,m,n} (\lambda_{n,k}-1)^{\pm\frac{1}{2}} |v_{k,l,m,n}|^2, \quad \Sigma_M^{\pm\frac{1}{2}} := \left\{ v, \|v\|_{\Sigma_M^{\pm\frac{1}{2}}} < \infty \right\},$$

and if $M = 0$ and $v_{k,l,m,0} = 0$,

$$(IX.9) \quad \|v\|_{X_{\perp}^{\pm\frac{1}{2}}}^2 := \sum_{n=1}^{\infty} \sum_{k,l,m} \left(l+1 + \sqrt{\lambda_{n,k}-1} \right)^{\pm 1} |v_{k,l,m,n}|^2, \quad X_{\perp}^{\pm\frac{1}{2}} := \left\{ v, \|v\|_{X_{\perp}^{\pm\frac{1}{2}}} < \infty \right\},$$

$$(IX.10) \quad \|v\|_{\Sigma_{\perp}^{\pm\frac{1}{2}}}^2 := \sum_{n=1}^{\infty} \sum_{k,l,m} (\lambda_{n,k} - 1)^{\pm\frac{1}{2}} |v_{k,l,m,n}|^2, \quad \Sigma_{\perp}^{\pm\frac{1}{2}} := \left\{ v, \|v\|_{\Sigma_{\perp}^{\pm\frac{1}{2}}} < \infty \right\}.$$

Theorem IX.1. *If $M > 0$, the global Cauchy problem for equation (VI.1) is well posed in $X_M^{\frac{1}{2}} \times X_M^{-\frac{1}{2}}$ and for the equation (VI.3) in $\Sigma_M^{\frac{1}{2}} \times \Sigma_M^{-\frac{1}{2}}$. The wave operators $\Omega_{in(out)}$ can be uniquely extended to isomorphisms from $X_M^{\frac{1}{2}} \times X_M^{-\frac{1}{2}}$ onto $\Sigma_M^{\frac{1}{2}} \times \Sigma_M^{-\frac{1}{2}}$. The scattering operator S is an isometry on $\Sigma_M^{\frac{1}{2}} \times \Sigma_M^{-\frac{1}{2}}$.*

If $M = 0$, the global Cauchy problem for equation (VI.1) is well posed in $X_{\perp}^{\frac{1}{2}} \times X_{\perp}^{-\frac{1}{2}}$ and for the equation (VI.3) in $\Sigma_{\perp}^{\frac{1}{2}} \times \Sigma_{\perp}^{-\frac{1}{2}}$. The wave operators $\Omega_{in(out)}$ can be uniquely extended to isomorphisms from $X_{\perp}^{\frac{1}{2}} \times X_{\perp}^{-\frac{1}{2}}$ onto $\Sigma_{\perp}^{\frac{1}{2}} \times \Sigma_{\perp}^{-\frac{1}{2}}$. The scattering operator S is an isometry on $\Sigma_{\perp}^{\frac{1}{2}} \times \Sigma_{\perp}^{-\frac{1}{2}}$.

Proof. We know that $\lambda_{n,k} \geq M^2 + n^2 + 1$, hence we have $0 < \delta := \inf(\frac{\pi}{2}\sqrt{\lambda_{n,k} - 1})$. The solutions of the Cauchy problem are given by the series (VI.21) and (VI.6) that are converging in the corresponding spaces (VI.15) and (VI.16). Furthermore (VI.13) and (VI.14) assure that there exists $C > 1$ such that for any solution $v \in C^0(\mathbb{R}_t, X^1) \cap C^1(\mathbb{R}_t, X^0)$ of (VI.1), we have with $* = M, \perp$:

$$(IX.11) \quad \frac{1}{C} \|(v(0), \partial_t v(0))\|_{X_*^{\frac{1}{2}} \times X_*^{-\frac{1}{2}}}^2 \leq \|(v_{in/out}(0), \partial_t v_{in/out}(0))\|_{\Sigma_*^{\frac{1}{2}} \times \Sigma_*^{-\frac{1}{2}}}^2 \leq C \|(v(0), \partial_t v(0))\|_{X_*^{\frac{1}{2}} \times X_*^{-\frac{1}{2}}}^2.$$

Since the density of $X^1 \times X^0$ in $X_M^{\frac{1}{2}} \times X_M^{-\frac{1}{2}}$ if $M > 0$, and $X_{\perp}^1 \times X_{\perp}^0$ in $X_{\perp}^{\frac{1}{2}} \times X_{\perp}^{-\frac{1}{2}}$ if $M = 0$, is obvious, we conclude that $\Omega_{in/out}$ can be extended by continuity into isomorphisms. Finally (VI.11) assures that S is an isometry.

Q.E.D.

We now consider the massive fields in the Hawking wormhole. Given $M > 0$ we introduce for

$$(IX.12) \quad v(x, \Omega_2) = \sum_{k,l,m} v_{k,l,m} w_k(x) Y_{l,m}(\Omega_2), \quad v_{k,l,m} \in \mathbb{C},$$

$$(IX.13) \quad \|v\|_{H^{\pm\frac{1}{2},M}}^2 := \sum_{k,l,m} \left(l + 1 + \sqrt{\lambda_k}\right)^{\pm 1} |v_{k,l,m}|^2, \quad H^{\pm\frac{1}{2},M} := \left\{ v, \|v\|_{H^{\pm\frac{1}{2},M}} < \infty \right\},$$

$$(IX.14) \quad \|v\|_{\dot{H}^{\pm\frac{1}{2},M}}^2 := \sum_{k,l,m} \lambda_k^{\pm\frac{1}{2}} |v_{k,l,m}|^2, \quad \dot{H}^{\pm\frac{1}{2},M} := \left\{ v, \|v\|_{\dot{H}^{\pm\frac{1}{2},M}} < \infty \right\},$$

By the same method we obtain the

Theorem IX.2. *If $M > 0$, the global Cauchy problem for equation (VIII.2) is well posed in $H^{\frac{1}{2},M} \times H^{-\frac{1}{2},M}$ and for the equation (VIII.10) in $\dot{H}^{\frac{1}{2},M} \times \dot{H}^{-\frac{1}{2},M}$. The wave operators $\Omega_{in(out)}$ can be uniquely extended to isomorphisms from $H^{\frac{1}{2},M} \times H^{-\frac{1}{2},M}$ onto $\dot{H}^{\frac{1}{2},M} \times \dot{H}^{-\frac{1}{2},M}$. The scattering operator S is an isometry on $\dot{H}^{\frac{1}{2},M} \times \dot{H}^{-\frac{1}{2},M}$.*

We now consider the massless case. A classic difficulty appears at the low energy, the so-called infrared problem, and we have to make a cut-off. In the sequel we fix $\delta > 0$. We begin with the Witten space with $M = 0$ and $\partial_{\psi} v = 0$. We write

$$(IX.15) \quad v(x, \Omega_2, \psi) = \left(\frac{1}{\sinh(2x)} \right)^{\frac{1}{2}} \sum_{l,m} Y_{l,m}(\Omega_2) \int_{1+\delta}^{\infty} \hat{v}_{l,m}(\lambda) \tanh\left(\frac{\pi}{2}\sqrt{\lambda-1}\right) Q_{-\frac{1}{2}}^{\frac{i}{2}\sqrt{\lambda-1}}(\coth(2x)) d\lambda,$$

we introduce

$$(IX.16) \quad \|v\|_{X_\delta^{\pm\frac{1}{2}}}^2 := \sum_{l,m} \int_{1+\delta}^{\infty} (l+1+\sqrt{\lambda})^{\pm 1} |\hat{v}_{l,m}(\lambda)|^2 d\lambda,$$

$$(IX.17) \quad \|v\|_{\dot{X}_\delta^{\pm\frac{1}{2}}}^2 := \sum_{l,m} \int_{1+\delta}^{\infty} \lambda^{\pm\frac{1}{2}} |\hat{v}_{l,m}(\lambda)|^2 d\lambda,$$

and we define $X_\delta^{\pm\frac{1}{2}}$ (respect. $\dot{X}_\delta^{\pm\frac{1}{2}}$), as the closure for the norm (IX.16) (respect. (IX.17)) of the set of functions v given by (IX.15) if $\hat{v}_{l,m} \in C_0^\infty(\mathbb{R})$ and $\hat{v}_{l,m} = 0$ for l large enough.

Theorem IX.3. *If $M = 0$, the global Cauchy problem for equation (VI.1) is well posed in $X_\delta^{\frac{1}{2}} \times X_\delta^{-\frac{1}{2}}$ and for the equation (VI.3) in $\dot{X}_\delta^{\frac{1}{2}} \times \dot{X}_\delta^{-\frac{1}{2}}$. The wave operators $\Omega_{in(out)}$ can be uniquely extended to isomorphisms from $X_\delta^{\frac{1}{2}} \times X_\delta^{-\frac{1}{2}}$ onto $\dot{X}_\delta^{\frac{1}{2}} \times \dot{X}_\delta^{-\frac{1}{2}}$. The scattering operator S is an isometry on $\dot{X}_\delta^{\frac{1}{2}} \times \dot{X}_\delta^{-\frac{1}{2}}$.*

Proof. As previously the solutions of the Cauchy problem are given by the series (VI.22) and (VI.24) that are converging in the corresponding spaces of interest. Furthermore, thanks to the cut-off $\delta > 0$, (VI.13) and (VI.14) assure that there exists $C_\delta > 1$ such that for any solutions $v \in C^0(\mathbb{R}_t, X_0^1) \cap C^1(\mathbb{R}_t, X_0^0)$ of (VI.1), we have:

$$(IX.18) \quad \frac{1}{C_\delta} \|(v(0), \partial_t v(0))\|_{X_\delta^{\frac{1}{2}} \times X_\delta^{-\frac{1}{2}}}^2 \leq \|(v_{in/out}(0), \partial_t v_{in/out}(0))\|_{\dot{X}_\delta^{\frac{1}{2}} \times \dot{X}_\delta^{-\frac{1}{2}}}^2 \leq C_\delta \|(v(0), \partial_t v(0))\|_{X_\delta^{\frac{1}{2}} \times X_\delta^{-\frac{1}{2}}}^2.$$

By the usual argument of continuity and density, we conclude that $\Omega_{in/out}$ can be extended by continuity into isomorphisms. Finally (VI.11) assures that S is an isometry again.

Q.E.D.

Finally we investigate the scattering of the massless fields in the Hawking wormhole. As explained previously, we do not consider the trivial case of the spherically symmetric fields, hence $l \geq 1$ in the expansions in spherical harmonics

$$(IX.19) \quad v(x, \Omega_2) = \sum_{l=1}^{\infty} \sum_{m=-l}^{m=l} v_{l,m}(x) Y_{l,m}(\Omega_2).$$

We introduce

$$(IX.20) \quad \|v\|_{H_\delta^{\pm\frac{1}{2}}}^2 := \sum_{l=1}^{\infty} \sum_{m=-l}^{m=l} \int_{-\infty}^{\infty} (l+|\xi|)^{\pm\frac{1}{2}} |\hat{v}_{l,m}(\xi)|^2 d\xi,$$

$$(IX.21) \quad \|v\|_{\dot{H}_\delta^{\pm\frac{1}{2}}}^2 := \sum_{l=1}^{\infty} \sum_{m=-l}^{m=l} \int_{-\infty}^{\infty} |\xi|^{\pm\frac{1}{2}} |\hat{v}_{l,m}(\xi)|^2 d\xi,$$

and the spaces $H_\delta^{\pm\frac{1}{2}}$ and $\dot{H}_\delta^{\pm\frac{1}{2}}$ defined as the closure of the set of functions v given by (IX.19) with $v_{l,m} = 0$ for l large enough, and $\hat{v}_{l,m} \in C_0^\infty(\mathbb{R} \setminus [-\delta, \delta])$. As previously we easily can prove the

Theorem IX.4. *If $M = 0$, the global Cauchy problem for equation (VIII.2) is well posed in $H_\delta^{\frac{1}{2}} \times H_\delta^{-\frac{1}{2}}$ and for the equation (VIII.10) in $\dot{H}_\delta^{\frac{1}{2}} \times \dot{H}_\delta^{-\frac{1}{2}}$. The wave operators $\Omega_{in(out)}$ can be uniquely extended to isomorphisms from $H_\delta^{\frac{1}{2}} \times H_\delta^{-\frac{1}{2}}$ onto $\dot{H}_\delta^{\frac{1}{2}} \times \dot{H}_\delta^{-\frac{1}{2}}$. The scattering operator S is an isometry on $\dot{H}_\delta^{\frac{1}{2}} \times \dot{H}_\delta^{-\frac{1}{2}}$.*

We emphasize that the Hawking wormhole is traversable by the massless fields: given $\epsilon = \pm 1$, if $\partial_t v_{in}(0, \cdot) = \epsilon v_{in}(0, \cdot)$ and v_{in} is supported in $\pm x > 0$, then $\partial_t v_{out}(0, \cdot) = \epsilon v_{out}(0, \cdot)$ and we conclude that the field v is asymptotically zero in the sheet $\pm x > 0$ as $t \rightarrow \infty$.

We now investigate the quantum fields. To be able to treat simultaneously the Witten space and the Hawking wormhole, we describe the common features of the scattering operators in both situations. S is an isometry on an asymptotic Hilbert space $\mathcal{E} = \Sigma_M^{\frac{1}{2}} \times \Sigma_M^{-\frac{1}{2}}, \Sigma_{\perp}^{\frac{1}{2}} \times \Sigma_{\perp}^{-\frac{1}{2}}, \dot{H}^{\frac{1}{2}, M} \times \dot{H}^{-\frac{1}{2}, M}, \dot{X}_{\delta}^{\frac{1}{2}} \times \dot{X}_{\delta}^{-\frac{1}{2}}, \dot{H}_{\delta}^{\frac{1}{2}} \times \dot{H}_{\delta}^{-\frac{1}{2}}$. Thanks to the spectral representations, we identify v with $(v_{k,l,m,n}), (v_{k,l,m}), (\hat{v}_{l,m}(\lambda))$, that we denote symbolically by $(v_{l,m}(\lambda))$. Then each space has the form

$$\mathcal{E} = L^2(\mathcal{I} \times \Lambda; |\lambda|^{\frac{1}{2}} \delta_{l,m} \otimes d_{\Lambda}) \times L^2(\mathcal{I} \times \Lambda; |\lambda|^{-\frac{1}{2}} \delta_{l,m} \otimes d_{\Lambda})$$

where $\delta_{l,m}$ is the counting measure on $\mathcal{I} = \{(l, m), 0 \text{ or } 1 \leq l, |m| \leq l\}$. For the massive case d_{Λ} is the counting measure on $\Lambda = \sigma(A)$, and for the massless case d_{Λ} is the Lebesgue measure on $\Lambda = [0, \infty[$ or $[\delta, \infty[$. We remark that these L^2 spaces are in duality, and there is a canonical symplectic form defined by:

$$(IX.22) \quad \sigma \left(\begin{pmatrix} v_1 \\ v'_1 \end{pmatrix}; \begin{pmatrix} v_2 \\ v'_2 \end{pmatrix} \right) = \int_{\mathcal{I} \times \Lambda} v_1 v'_2 - v'_1 v_2 \delta_{l,m} d_{\Lambda}.$$

We split \mathcal{E} as an orthogonal sum of spaces of the positive and negative frequency particles:

$$(IX.23) \quad \mathcal{E} = \mathcal{E}_{pos} \oplus \mathcal{E}_{neg}, \quad (v, v') \in \mathcal{E}_{pos(neg)} \Leftrightarrow v' = +(-)i |\lambda|^{\frac{1}{2}} v.$$

In this representation, the propagator associating $(v(t), \partial_t v(t))$ to $(v(0), \partial_t v(0))$, v being solution of the asymptotic equations (VI.3) and (VIII.10), leaves invariant these spaces and acts simply as

$$(IX.24) \quad (v, v') \in \mathcal{E}_{pos(neg)}, \quad U(t)(v, v') = e^{+(-)i|\lambda|^{\frac{1}{2}}t}(v, v').$$

The crucial point is that the scattering operator does not mix the particles and the antiparticles since if $(v_{in}, v'_{in}) \in \mathcal{E}_{pos(neg)}$, we have

$$(IX.25) \quad v_{out;l,m}(\lambda) = (-1)^l \frac{\Gamma(1 + (-)i\sqrt{\lambda})}{\Gamma(1 - (+)i\sqrt{\lambda})} \frac{\Gamma(l + 1 - (+)i\sqrt{\lambda})}{\Gamma(l + 1 + (-)i\sqrt{\lambda})} v_{in;l,m}(\lambda), \quad v'_{out;l,m}(\lambda) = +(-)i |\lambda|^{\frac{1}{2}} v_{in;l,m}$$

so we get:

$$(IX.26) \quad (v_{in}, v'_{in}) \in \mathcal{E}_{pos(neg)}, \quad (v_{out}, v'_{out}) = S((v_{in}, v'_{in})) \in \mathcal{E}_{pos(neg)}.$$

Finally (IX.24) and (IX.25) assure that $U(t)$ and S are isometries commuting on $\mathcal{E}_{pos(neg)}$, and they are symplectic. We have all the ingredients to apply the functorial machinery of the second quantization (see. e.g. [24] or [26]) and we can use the method employed in [4] for the Schwarzschild Black-Hole, based on the uniqueness result of Kay [38]. We obtain the:

Theorem IX.5. *The scattering operator S is unitarily implementable in the Fock-Cook quantization of (\mathcal{E}, U, σ) , and the quantized scattering operator leaves invariant the Fock vacuum.*

At first glance, this result is very surprising because the metrics are deeply time-dependent, and we could believe that some creation of particles appears in the dynamical spacetimes (see e.g. [9]). The root of this result is the coefficient of reflection being zero. In fact, the existence of the Kaluza-Klein towers shows that the dynamics in the Bubble of nothing of Witten and in the Hawking wormhole, mainly obey to the dynamics in dS^3 . Therefore the behaviour of the fields is governed by the Schrödinger equation (V.15) with the Pöschl-Teller potential $\lambda(\lambda + 1)/\cosh^2 t$. This potential appears in various contexts such as the vibrational excitations of diatomic molecules, the Korteweg-de Vries equation and the plane stratified dielectric medium [39]. We have known for a long time that this potential is reflectionless iff λ is an integer. In the case of the 3-dimensional

De Sitter space, we have $\lambda = l$, where we have used the spherical harmonics expansion with $\Delta_{S^2} Y_{l,m} = -l(l+1)Y_{l,m}$, $l \in \mathbb{N}$. In the general case of a d -dimensional De Sitter spacetime, we have $\Delta_{S^{d-1}} Y_{l,m} = -l(l+d-2)Y_{l,m}$ and $\lambda = \frac{2l+d-3}{2}$. We conclude that the origin of our result is the quantum transparency of the odd-dimensional De Sitter spaces. This property has been noted in [13]. We emphasize that our result deals only with the Fock vacuum states for the asymptotic dynamics. In the massless case this vacuum is analogous to the usual Fock vacuum at the null infinities \mathcal{I}_\pm of the Minkowski space, and in the massive case it is analogous to the Bunch-Davies state on the De Sitter spacetime dS^3 investigated in [2], [16], [19]. Therefore our work leaves open several important issues concerning the quantum states on the Witten spacetime and the Hawking wormhole, such as an explicit expression of the two-point function, the Hadamard property, the classification of other interesting vacuum states.

We end this part by returning to the classical fields and we make some remarks on the notion of resonances in our context. Since the Witten and Hawking metrics are not stationnary, the Hamiltonian of Klein-Gordon is time-dependent and we cannot define the resonances as the poles of the meromorphic continuation of the resolvent operator. Nevertheless, we can take advantage of the existence of a continuous spectral parameter in the scattering operator in the massless case, and we here address the issue of its analytic continuation. The scattering amplitude is given by (IX.25):

$$(IX.27) \quad (-1)^l \frac{\Gamma(1 + (-)i\sqrt{\lambda}) \Gamma(l+1 - (+)i\sqrt{\lambda})}{\Gamma(1 - (+)i\sqrt{\lambda}) \Gamma(l+1 + (-)i\sqrt{\lambda})}.$$

If we consider this quantity as a complex function of the complex variable $\sqrt{\lambda}$, it admits a meromorphic continuation on \mathbb{C} with simple poles for

$$(IX.28) \quad \sqrt{\lambda} = +(-)(n+1)i, \quad 0 \leq n < l.$$

We remark that for these values, we have

$$(IX.29) \quad \left| P_l^{i\sqrt{\lambda}}(\pm \tanh t) \right| \lesssim (\cosh t)^{-n-1}.$$

Since

$$(IX.30) \quad v(t, x, \Omega_2) = P_l^{\pm i(n+1)}(\tanh t) Y_{l,m}(\Omega_2) Q_{-\frac{1}{2}}^{\pm \frac{n+1}{2}}(\coth(2x))$$

is solution of (VI.1) and

$$(IX.31) \quad v(t, x, \Omega_2) = P_l^{\pm i(n+1)}(\tanh t) Y_{l,m}(\Omega_2) e^{\pm(n+1)x}$$

is solution of (VIII.2), we conclude that for a scattering resonance, there are profiles that are disappearing as $|t| \rightarrow \infty$. The squared masses of these fields are negative and take an infinite set of discrete values $\lambda = -(n+1)^2$. These solutions of the Klein-Gordon equation on the Witten spacetime are analogous to the tachyons on the De Sitter space investigated in [28]. It would be interesting to investigate the problem of the resonances on the Witten spacetime with the powerful technics developped for the asymptotically De Sitter/Minkowski spaces in [7], [49], [50]. In this context, the resonances are the poles of the inverses of a family of operators constructed with the Mellin transform. We let open the issue determining if the Witten spacetime and the Hawking wormhole fit into this framework.

X. CONCLUSION AND OPEN ISSUES

In this work we have considered the Klein-Gordon equation in the Witten spacetime and in its remarkable submanifold: the Lorentzian Hawking wormhole. Taking advantage of the global hyperbolicity of these spacetimes, we have solved the global Cauchy problem in the functional framework associated with the energy. Performing a complete spectral analysis of the Hamiltonians, we have

obtained analytic expressions of the fields as Kaluza-Klein towers involving the solutions of the Klein-Gordon equation on the 2+1 dimensional De Sitter spacetime. We have deduced the asymptotic behaviours of the profiles $v(t) = \cosh(t)u(t)$ of a wave u as time tends to infinity: in the massive case, v is asymptotically quasi periodic, and in the massless case v is dispersive. Therefore the massive fields stay localised near the bubble of nothing, or near the throat of the wormhole. In contrast, the wormhole is traversable by the massless fields. We have constructed the scattering operator linking the asymptotic fields to the past and future infinities (null infinity in the massless case, timelike infinity in the massive case). The quantized scattering operator leaves invariant the Fock vacuum: there is no creation of particle.

We end this paper by evoking several open problems. It would be interesting to investigate the scalar waves in more complicated contexts involving one or several bubbles of nothing. For instance an exact solution describing the collision of two bubbles of nothing is constructed in [36] and general configurations of charged and static black holes sitting on a bubble are presented in [40]. By analytic continuation of the fifth dimensional Kerr metric, we obtain a rotating bubble of nothing [1], [27]. This bubble behaves qualitatively differently from the spherical bubble of Witten: the compact dimension opens up asymptotically, while in the Witten spacetime it does not. We can expect that these properties have interesting consequences for the behaviour of the fields and the scattering theory. The study of the nonlinear stability of these solutions of the Einstein equations is a very difficult open problem. As we have mentioned, it would be natural to pursue the study of the quantum states. For instance we could wonder whether the Sorkin-Johnston formalism used in [3] to determine a preferred ground state in the De Sitter space could be applied to the Witten universe. Finally we remark that in physics there are two concepts of “nothing” as absence of spacetime: the first one, that is the purpose of this paper, is the bubble of nothing as an endpoint of tunneling, and the second one is the nothing as a starting point from which the universe can tunnel: this is the fascinating concept of quantum creation of the universe from nothing (see *e.g.* [11] and the references therein). These two tunnelings are treated within a unified framework in [17]. We hope that our work will be useful to perform a mathematically rigorous approach for this quantum cosmogony.

APPENDIX

In this appendix, we present the Christoffel symbols of the Witten metric, computed with several coordinates. Outside the bubble of nothing $\rho = R$, we use the Schwarzschild type coordinates $(t, \rho, \theta, \varphi, \psi)$, for which the metric is given by:

$$ds_{Witten}^2 = \rho^2 dt^2 - \left(1 - \frac{R^2}{\rho^2}\right)^{-1} d\rho^2 - \rho^2 \cosh^2 t \left(d\theta^2 + \sin^2 \theta d\varphi^2\right) - \left(1 - \frac{R^2}{\rho^2}\right) d\psi^2.$$

The non zero Christoffel symbols are:

$$\begin{aligned} \Gamma_{t\rho}^t &= \Gamma_{\rho t}^t = \frac{1}{\rho}, \quad \Gamma_{\theta\theta}^t = \sinh t \cosh t, \quad \Gamma_{\varphi\varphi}^t = \sinh t \cosh t \sin^2 \theta, \\ \Gamma_{tt}^\rho &= \rho \left(1 - \frac{R^2}{\rho^2}\right), \quad \Gamma_{\rho\rho}^\rho = -\frac{R^2}{\rho^3} \left(1 - \frac{R^2}{\rho^2}\right)^{-1}, \quad \Gamma_{\theta\theta}^\rho = -\rho \left(1 - \frac{R^2}{\rho^2}\right) \cosh^2 t, \\ \Gamma_{\varphi\varphi}^\rho &= -\rho \left(1 - \frac{R^2}{\rho^2}\right) \cosh^2 t \sin^2 \theta, \quad \Gamma_{\psi\psi}^\rho = -\frac{R^2}{\rho^3} \left(1 - \frac{R^2}{\rho^2}\right), \\ \Gamma_{\theta t}^\theta &= \Gamma_{t\theta}^\theta = \frac{\sinh t}{\cosh t}, \quad \Gamma_{\rho\theta}^\theta = \Gamma_{\theta\rho}^\theta = \frac{1}{\rho}, \quad \Gamma_{\varphi\varphi}^\theta = -\sin \theta \cos \theta, \\ \Gamma_{t\varphi}^\varphi &= \Gamma_{\varphi t}^\varphi = \frac{\sinh t}{\cosh t}, \quad \Gamma_{\rho\varphi}^\varphi = \Gamma_{\varphi\rho}^\varphi = \frac{1}{\rho}, \quad \Gamma_{\theta\varphi}^\varphi = \Gamma_{\varphi\theta}^\varphi = \frac{\cos \theta}{\sin \theta}, \end{aligned}$$

$$\Gamma_{\rho\psi}^\psi = \Gamma_{\psi\rho}^\psi = \frac{R^2}{\rho^3} \left(1 - \frac{R^2}{\rho^2}\right)^{-1}.$$

In the $(t, y, z, \omega) \in \mathbb{R} \times \mathbb{R}^2 \times S^2$ coordinates, for which the Witten metric is (we take $R = 1$)

$$ds_{Witten}^2 = \rho^2 dt^2 - \frac{(1 + \rho)^2}{\rho^2} e^{-2\rho} (dy^2 + dz^2) - \rho^2 \cosh^2 t d\Omega_2^2,$$

and ρ is the C^∞ function of (y, z) given by the generalized Lambert function [6],

$$\rho = \frac{1}{2} W \left(\frac{+2}{-2}, y^2 + z^2 \right),$$

the non zero Christoffel symbols are:

$$\Gamma_{ty}^t = \Gamma_{yt}^t = \frac{y}{\rho} W' \left(\frac{+2}{-2}, y^2 + z^2 \right), \quad \Gamma_{tz}^t = \Gamma_{zt}^t = \frac{z}{\rho} W' \left(\frac{+2}{-2}, y^2 + z^2 \right),$$

$$\Gamma_{\theta\theta}^t = \sinh t \cosh t, \quad \Gamma_{\varphi\varphi}^t = \sinh t \cosh t \sin^2 \theta,$$

$$\Gamma_{tt}^y = y\rho \left(1 + \frac{1}{\rho}\right)^{-2} e^{2\rho} W' \left(\frac{+2}{-2}, y^2 + z^2 \right), \quad \Gamma_{tt}^z = z\rho \left(1 + \frac{1}{\rho}\right)^{-2} e^{2\rho} W' \left(\frac{+2}{-2}, y^2 + z^2 \right),$$

$$\Gamma_{yy}^y = -\Gamma_{zz}^y = \Gamma_{yz}^z = \Gamma_{zy}^z = -y \left(1 + \frac{1}{\rho}\right)^{-1} \left(1 + \frac{1}{\rho} + \frac{1}{\rho^2}\right) W' \left(\frac{+2}{-2}, y^2 + z^2 \right),$$

$$\Gamma_{yz}^y = \Gamma_{zy}^y = \Gamma_{zz}^z = -\Gamma_{yy}^z = -z \left(1 + \frac{1}{\rho}\right)^{-1} \left(1 + \frac{1}{\rho} + \frac{1}{\rho^2}\right) W' \left(\frac{+2}{-2}, y^2 + z^2 \right),$$

$$\Gamma_{\theta\theta}^y = -y\rho \left(1 + \frac{1}{\rho}\right)^{-2} e^{2\rho} \cosh^2 t W' \left(\frac{+2}{-2}, y^2 + z^2 \right), \quad \Gamma_{\varphi\varphi}^y = \Gamma_{\theta\theta}^y \sin^2 \theta,$$

$$\Gamma_{\theta\theta}^z = -z\rho \left(1 + \frac{1}{\rho}\right)^{-2} e^{2\rho} \cosh^2 t W' \left(\frac{+2}{-2}, y^2 + z^2 \right), \quad \Gamma_{\varphi\varphi}^z = \Gamma_{\theta\theta}^z \sin^2 \theta,$$

$$\Gamma_{\theta t}^\theta = \Gamma_{t\theta}^\theta = \Gamma_{t\varphi}^\varphi = \Gamma_{\varphi t}^\varphi = \frac{\sinh t}{\cosh t}, \quad \Gamma_{\varphi\varphi}^\theta = -\sin \theta \cos \theta,$$

$$\Gamma_{\theta y}^\theta = \Gamma_{y\theta}^\theta = \Gamma_{\varphi y}^\varphi = \Gamma_{y\varphi}^\varphi = \frac{y}{\rho} W' \left(\frac{+2}{-2}, y^2 + z^2 \right), \quad \Gamma_{\theta z}^\theta = \Gamma_{z\theta}^\theta = \Gamma_{\varphi z}^\varphi = \Gamma_{z\varphi}^\varphi = \frac{z}{\rho} W' \left(\frac{+2}{-2}, y^2 + z^2 \right).$$

INDEX OF NOTATION

H^0	(V.4)	$H_\delta^{\pm\frac{1}{2}}$	(IX.20)	W_0^1	(IV.10)	$X_M^{\pm\frac{1}{2}}$	(IX.7)
H_0^0	(V.4)	$\dot{H}_\delta^{\pm\frac{1}{2}}$	(IX.21)	W_\perp^1	(IV.10)	$X_\perp^{\pm\frac{1}{2}}$	(IX.9)
H_\perp^0	(V.4)	$H^{1,M}$	(VIII.4)	X_\perp^0	(IV.8)	X_\perp^1	(IV.17)
H_n^0	(V.22)	$\dot{H}^{1,M}$	(VIII.11)	X_0^0	(IV.10)	X_0^1	(IV.17)
\mathfrak{h}_0	(V.29)	$\Sigma_M^{\pm\frac{1}{2}}$	(IX.8)	X_\perp^0	(IV.10)	X_\perp^1	(IV.17)
$H^{\pm\frac{1}{2},M}$	(IX.13)	$\Sigma_\perp^{\pm\frac{1}{2}}$	(IX.10)	$X_\delta^{\pm\frac{1}{2}}$	(IX.16)	\dot{X}^1	(VI.5)
$\dot{H}^{\pm\frac{1}{2},M}$	(IX.14)	W^1	(IV.9)	$\dot{X}_\delta^{\pm\frac{1}{2}}$	(IX.17)	Y^1	(IV.8)

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